Sets

Section 2.1

Section Summary

- Definition of sets
- Describing Sets
  - Roster Method
  - Set-Builder Notation
- Some Important Sets in Mathematics
- Empty Set and Universal Set
- Subsets and Set Equality
- Cardinality of Sets
- Tuples
- Cartesian Products

Introduction

- Sets are one of the basic building blocks for the types of objects considered in discrete mathematics.
  - Important for counting.
  - Programming languages have set operations.
- Set theory is an important branch of mathematics.
  - Many different systems of axioms have been used to develop set theory.
  - Here we are not concerned with a formal set of axioms for set theory. Instead, we will use what is called naïve set theory.

Sets

- A *set* is an unordered collection of objects.
  - the students in this class
  - the chairs in this room
- The objects in a set are called the *elements*, or *members* of the set. A set is said to *contain* its elements.
- The notation $a \in A$ denotes that a is an element of the set A.
- If a is not a member of A, write $a \notin A$

Describing a Set: Roster Method

- $S = \{a,b,c,d\}$
- Order not important

$$S = \{a,b,c,d\} = \{b,c,a,d\}$$

- Each distinct object is either a member or not; listing more than once does not change the set.

$$S = \{a,b,c,d\} = \{a,b,c,b,c,d\}$$

- Elipses (…) may be used to describe a set without listing all of the members when the pattern is clear.

$$S = \{a,b,c,d, \ldots\ldots,z\}$$

Roster Method

- Set of all vowels in the English alphabet:

  $V = \{a,e,i,o,u\}$

- Set of all odd positive integers less than 10:

  $O = \{1,3,5,7,9\}$

- Set of all positive integers less than 100:

  $S = \{1,2,3,……..,99\}$

- Set of all integers less than 0:

  $S = \{…., -3,-2,-1\}$

Some Important Sets

N = *natural numbers* = {0,1,2,3….}

Z = *integers* = {…,-3,-2,-1,0,1,2,3,…}

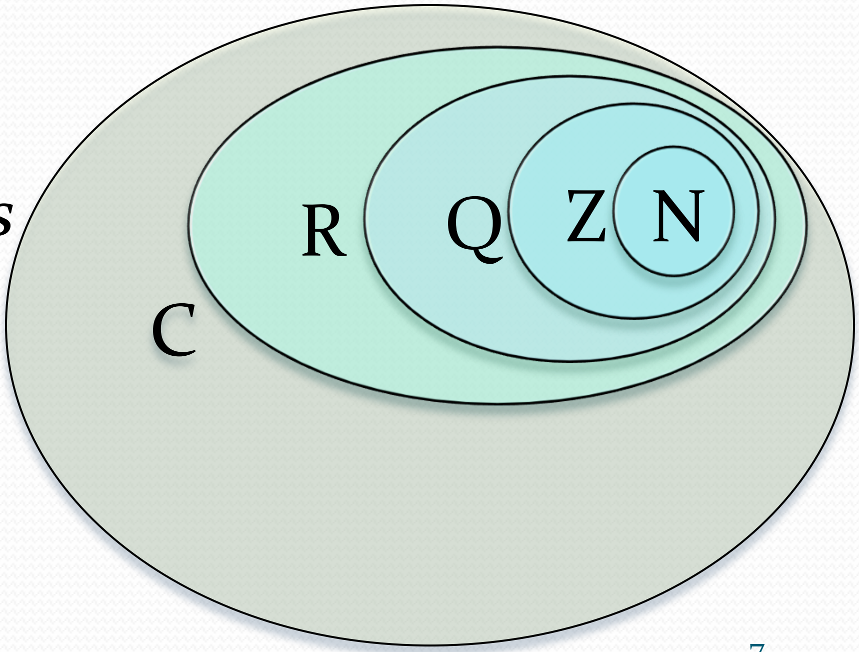
$\mathbf{Z}^{+}$ = *positive integers* = {1,2,3,…..}

Q = set of rational numbers

R = set of *real numbers*

$\mathbf{R}^{+}$ = set of *positive real numbers*

C = set of *complex numbers*.



Set-Builder Notation

- Specify the property or properties that all members must satisfy:

  $S = \{x \mid x \text{ is a positive integer less than } 100\}$

  $O = \{x \mid x \text{ is an odd positive integer less than } 10\}$

  $O = \{x \in \mathbf{Z}^{+} \mid x \text{ is odd and } x < 10\}$

- A predicate may be used:

  $$S = \{x \mid \mathrm{P}(x)\}$$

- Example: $S = \{x \mid \mathrm{Prime}(x)\}$
- Positive rational numbers:

  $$\mathbf{Q}^{+} = \{x \in \mathbf{R} \mid x = p/q, \text{ for some positive integers } p,q\}$$

Interval Notation

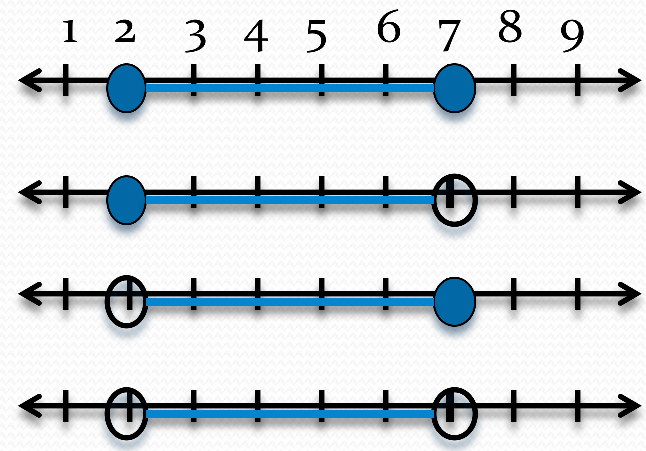
$$[a,b] = \{x \mid a \le x \le b\}$$
$$[a,b) = \{x \mid a \le x < b\}$$
$$(a,b] = \{x \mid a < x \le b\}$$
$$(a,b) = \{x \mid a < x < b\}$$

[2, 7]

[2, 7)

(2, 7]

(2, 7)



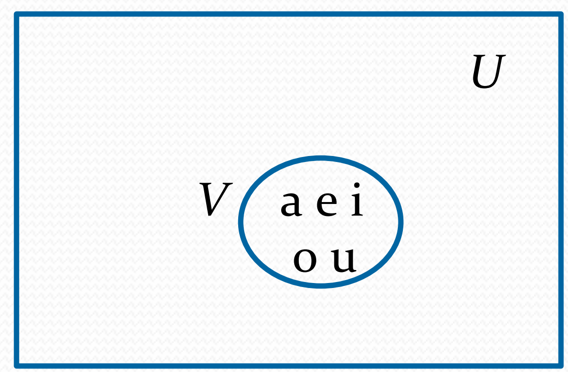
closed interval [a,b]

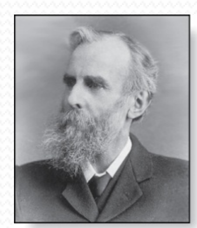
open interval (a,b)

Universal Set and Empty Set

- The *universal set U* is the set containing everything currently under consideration.
  - Sometimes implicit
  - Sometimes explicitly stated.
  - Contents depend on the context.
- The empty set is the set with no elements. Symbolized ∅, but {} also used.

Venn Diagram

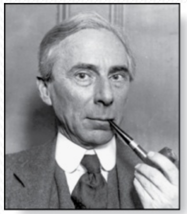




John Venn (1834-1923)
Cambridge, UK

Russell's Paradox

- Let S be the set of all sets which are not members of themselves. A paradox results from trying to answer the question "Is S a member of itself?"
- Related Paradox:
  - Henry is a barber who shaves all people who do not shave themselves. A paradox results from trying to answer the question "Does Henry shave himself?"



Bertrand Russell (1872-1970)
Cambridge, UK
Nobel Prize Winner

Some things to remember

- Sets can be elements of sets.

  $\{\{1,2,3\},a, \{b,c\}\}$

  $\{\mathbf{N},\mathbf{Z},\mathbf{Q},\mathbf{R}\}$

- The empty set is different from a set containing the empty set.

  $\emptyset \neq \{ \emptyset \}$

Set Equality

Definition: Two sets are *equal* if and only if they have the same elements.

- Therefore if A and B are sets, then A and B are equal if and only if $\forall x(x \in A \leftrightarrow x \in B)$
- We write $A = B$ if A and B are equal sets.

$$\{1,3,5\} = \{3, 5, 1\}$$

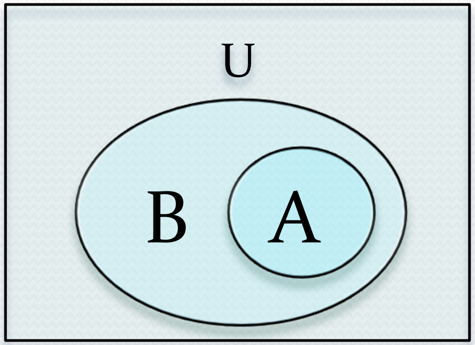
$$\{1,5,5,5,3,3,1\} = \{1,3,5\}$$

Subsets

Definition: The set A is a *subset* of B, if and only if every element of A is also an element of B.

- The notation $A \subseteq B$ is used to indicate that A is a subset of the set B.
- $A \subseteq B$ holds if and only if $\forall x(x \in A \to x \in B)$ is true.
  1. Because $a \in \emptyset$ is always false, $\emptyset \subseteq S$, for every set S.
  2. Because $a \in S \to a \in S$, $S \subseteq S$, for every set S.

$A \subseteq B$



Showing a Set is or is not a Subset of Another Set

- **A is a Subset of B**: To show $A \subseteq B$, show that if x belongs to A, then x also belongs to B.
- **A is not a Subset of B**: To show $A \nsubseteq B$, find an element $x \in A$ such that $x \notin B$. (Such an x is a counterexample to the claim that $x \in A$ implies $x \in B$.)

Examples:

1. The set of all computer science majors at your school is a subset of all students at your school.
2. The set of integers with squares less than 100 is not a subset of the set of nonnegative integers.

Another look at Equality of Sets

- Recall that two sets A and B are *equal*, denoted by $A = B$, iff $\forall x(x \in A \leftrightarrow x \in B)$

- Using logical equivalences we have that $A = B$ iff
$$\forall x[(x \in A \to x \in B) \land (x \in B \to x \in A)]$$

- This is equivalent to

$A \subseteq B$ and $B \subseteq A$

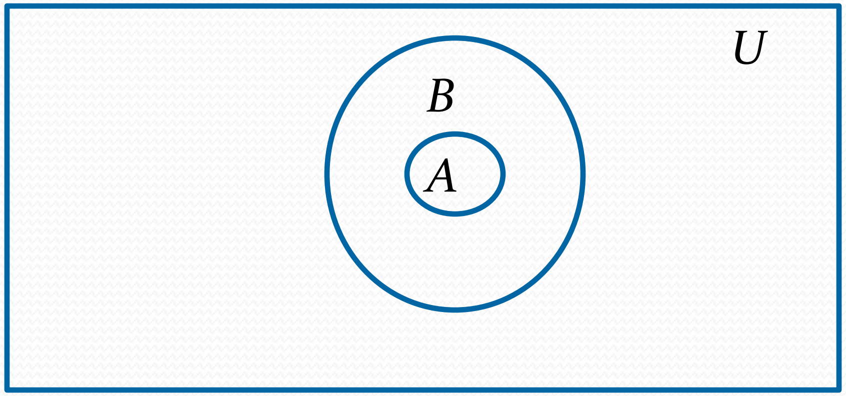
Proper Subsets

Definition: If $A \subseteq B$, but $A \neq B$, then we say A is a *proper subset* of B, denoted by $A \subset B$. If $A \subset B$, then

$$\forall x (x \in A \rightarrow x \in B) \wedge \exists x (x \in B \wedge x \notin A)$$

is true.

Venn Diagram



Set Cardinality

Definition: If there are exactly n distinct elements in S where n is a nonnegative integer, we say that S is *finite*. Otherwise it is *infinite*.

Definition: The *cardinality* of a finite set A, denoted by $|A|$, is the number of (distinct) elements of A.

Examples:

1. $|\varnothing| = 0$
2. Let S be the letters of the English alphabet. Then $|S| = 26$
3. $|\{1,2,3\}| = 3$
4. $|\{\varnothing\}| = 1$
5. The set of integers is infinite.

Power Sets

Definition: The set of all subsets of a set A, denoted $P(A)$, is called the *power set* of A.

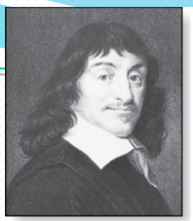
Ex: If $A = \{a,b\}$ then

$$\mathcal{P}(A) = \{\emptyset, \{a\},\{b\},\{a,b\}\}$$

- If a set has n elements, then the cardinality of the power set is 2^n. (In Ch. 5 and 6, we'll discuss different ways to show this.)

Tuples

- The *ordered n-tuple* $(a_1, a_2, \ldots, a_n)$ is the ordered collection that has a_1 as its first element and a_2 as its second element and so on until a_n as its last element.
- Two n-tuples are equal if and only if their corresponding elements are equal.
- 2-tuples are called *ordered pairs*.
- The ordered pairs (a,b) and (c,d) are equal if and only if $a = c$ and $b = d$.



René Descartes
(1596-1650)

Cartesian Product

Definition: The *Cartesian Product* of two sets A and B, denoted by $A \times B$ is the set of ordered pairs (a,b) where $a \in A$ and $b \in B$.

$$A \times B = \{(a,b) | a \in A \wedge b \in B\}$$

Example:

$A = \{a,b\}$ $B = \{1,2,3\}$

$A \times B = \{(a,1),(a,2),(a,3), (b,1),(b,2),(b,3)\}$

$B \times \mathrm{A} = \{(1,\mathrm{a}),(1,\mathrm{b}), (2,\mathrm{a}),(2,\mathrm{b}), (3,\mathrm{a}),(3,\mathrm{b})\}$

- **Definition**: A subset R of the Cartesian product $A \times B$ is called a *relation* from the set A to the set B. (More in Chapter 9.)

Cartesian Product

Definition: The cartesian products of the sets $A_1, A_2, \ldots\ldots, A_n$, denoted by $A_1 \times A_2 \times \ldots\ldots \times A_n$, is the set of ordered n-tuples $(a_1, a_2, \ldots\ldots, a_n)$ where a_i belongs to A_i for $i = 1, \ldots n$.

$$A_1 \times A_2 \times \cdots \times A_n = \{(a_1, a_2, \ldots, a_n) | a_i \in A_i \text{ for } i = 1, 2, \ldots n\}$$

Ex: What is $A \times B \times C$ where $A = \{0,1\}$, $B = \{1,2\}$ and $C = \{0,1,2\}$

Solution: $A \times B \times C = \{(0,1,0), (0,1,1), (0,1,2), (0,2,0), (0,2,1), (0,2,2), (1,1,0), (1,1,1), (1,1,2), (1,2,0), (1,2,1), (1,2,2)\}$

Set Notation with Quantifiers

- $\forall x \in S\ (P(x))$ is shorthand for $\forall x(x \in S \rightarrow P(x))$
- $\exists x \in S\ (P(x))$ is shorthand for $\exists x(x \in S \wedge P(x))$

Ex: Express the following in English

1. $\forall x \in \mathbf{R}\ (x^2 \geq 0)$
   - "The square of every real number is nonnegative."
2. $\exists x \in \mathbf{Z}\ (x^2 = 1)$
   - "There is an integer whose square is one."

Truth Sets and Quantifiers

- Given a predicate P and a domain D, we define the *truth set* of P to be the set of elements in D for which $P(x)$ is true. The truth set of P(x) is denoted by

$$\{x \in D | P(x)\}$$

- **Ex**: The truth set of $P(x)$ where the domain is the integers and $P(x)$ is "$|x| = 1$" is the set {-1,1}