Discrete Structures Homework 6

Include your name, the homework number, and your complete work, including any steps used to obtain the answer. Submit a hard copy - written out legibly or printed - before class.

<u>Section 2.3</u>
Problems 4, 12, 22, 30, 36

<u>Section 2.4</u>
Problems 2, 10, 16, 26, 30

Partial Functions

A program designed to evaluate a function may not produce the correct value of the function for all elements in the domain of this function. For example, a program may not produce a correct value because evaluating the function may lead to an infinite loop or an overflow. Similarly, in abstract mathematics, we often want to discuss functions that are defined only for a subset of the real numbers, such as $1/x$, $\sqrt{x}$, and $\arcsin(x)$. Also, we may want to use such notions as the "youngest child" function, which is undefined for a couple having no children, or the "time of sunrise," which is undefined for some days above the Arctic Circle. To study such situations, we use the concept of a partial function.

DEFINITION 13 A *partial function* f from a set A to a set B is an assignment to each element a in a subset of A, called the *domain of definition* of f, of a unique element b in B. The sets A and B are called the *domain* and *codomain* of f, respectively. We say that f is *undefined* for elements in A that are not in the domain of definition of f. When the domain of definition of f equals A, we say that f is a *total function.*

Remark: We write $f : A \to B$ to denote that f is a partial function from A to B. Note that this is the same notation as is used for functions. The context in which the notation is used determines whether f is a partial function or a total function.

EXAMPLE 32 The function $f : \mathbf{Z} \to \mathbf{R}$ where $f(n) = \sqrt{n}$ is a partial function from $\mathbf{Z}$ to $\mathbf{R}$ where the domain of definition is the set of nonnegative integers. Note that f is undefined for negative integers. ◂

Exercises

Section 2.3

1. Why is f not a function from $\mathbf{R}$ to $\mathbf{R}$ if
   a) $f(x) = 1/x$?
   b) $f(x) = \sqrt{x}$?
   c) $f(x) = \pm\sqrt{(x^2+1)}$?
2. Determine whether f is a function from $\mathbf{Z}$ to $\mathbf{R}$ if
   a) $f(n) = \pm n$.
   b) $f(n) = \sqrt{n^2+1}$.
   c) $f(n) = 1/(n^2-4)$.
3. Determine whether f is a function from the set of all bit strings to the set of integers if
   a) $f(S)$ is the position of a 0 bit in S.
   b) $f(S)$ is the number of 1 bits in S.
   c) $f(S)$ is the smallest integer i such that the ith bit of S is 1 and $f(S) = 0$ when S is the empty string, the string with no bits.
4. Find the domain and range of these functions. Note that in each case, to find the domain, determine the set of elements assigned values by the function.
   a) the function that assigns to each nonnegative integer its last digit
   b) the function that assigns the next largest integer to a positive integer
   c) the function that assigns to a bit string the number of one bits in the string
   d) the function that assigns to a bit string the number of bits in the string
5. Find the domain and range of these functions. Note that in each case, to find the domain, determine the set of elements assigned values by the function.
   a) the function that assigns to each bit string the number of ones in the string minus the number of zeros in the string
   b) the function that assigns to each bit string twice the number of zeros in that string
   c) the function that assigns the number of bits left over when a bit string is split into bytes (which are blocks of 8 bits)
   d) the function that assigns to each positive integer the largest perfect square not exceeding this integer
6. Find the domain and range of these functions.
   a) the function that assigns to each pair of positive integers the first integer of the pair
   b) the function that assigns to each positive integer its largest decimal digit
   c) the function that assigns to a bit string the number of ones minus the number of zeros in the string
   d) the function that assigns to each positive integer the largest integer not exceeding the square root of the integer
   e) the function that assigns to a bit string the longest string of ones in the string

7. Find the domain and range of these functions.

a) the function that assigns to each pair of positive integers the maximum of these two integers

b) the function that assigns to each positive integer the number of the digits 0, 1, 2, 3, 4, 5, 6, 7, 8, 9 that do not appear as decimal digits of the integer

c) the function that assigns to a bit string the number of times the block 11 appears

d) the function that assigns to a bit string the numerical position of the first 1 in the string and that assigns the value 0 to a bit string consisting of all 0s

8. Find these values.

a) $\lfloor 1.1 \rfloor$ b) $\lceil 1.1 \rceil$
c) $\lfloor -0.1 \rfloor$ d) $\lceil -0.1 \rceil$
e) $\lceil 2.99 \rceil$ f) $\lceil -2.99 \rceil$
g) $\lfloor \frac{1}{2} + \lceil \frac{1}{2} \rceil \rfloor$ h) $\lceil \lfloor \frac{1}{2} \rfloor + \lceil \frac{1}{2} \rceil + \frac{1}{2} \rceil$

9. Find these values.

a) $\lceil \frac{3}{4} \rceil$ b) $\lfloor \frac{7}{8} \rfloor$
c) $\lceil -\frac{3}{4} \rceil$ d) $\lfloor -\frac{7}{8} \rfloor$
e) $\lceil 3 \rceil$ f) $\lfloor -1 \rfloor$
g) $\lfloor \frac{1}{2} + \lceil \frac{3}{2} \rceil \rfloor$ h) $\lfloor \frac{1}{2} \cdot \lfloor \frac{5}{2} \rfloor \rfloor$

10. Determine whether each of these functions from $\{a, b, c, d\}$ to itself is one-to-one.

a) $f(a) = b,\ f(b) = a,\ f(c) = c,\ f(d) = d$
b) $f(a) = b,\ f(b) = b,\ f(c) = d,\ f(d) = c$
c) $f(a) = d,\ f(b) = b,\ f(c) = c,\ f(d) = d$

11. Which functions in Exercise 10 are onto?

12. Determine whether each of these functions from $\mathbf{Z}$ to $\mathbf{Z}$ is one-to-one.

a) $f(n) = n - 1$ b) $f(n) = n^2 + 1$
c) $f(n) = n^3$ d) $f(n) = \lceil n/2 \rceil$

13. Which functions in Exercise 12 are onto?

14. Determine whether $f: \mathbf{Z} \times \mathbf{Z} \to \mathbf{Z}$ is onto if

a) $f(m, n) = 2m - n$.
b) $f(m, n) = m^2 - n^2$.
c) $f(m, n) = m + n + 1$.
d) $f(m, n) = |m| - |n|$.
e) $f(m, n) = m^2 - 4$.

15. Determine whether the function $f: \mathbf{Z} \times \mathbf{Z} \to \mathbf{Z}$ is onto if

a) $f(m, n) = m + n$.
b) $f(m, n) = m^2 + n^2$.
c) $f(m, n) = m$.
d) $f(m, n) = |n|$.
e) $f(m, n) = m - n$.

16. Consider these functions from the set of students in a discrete mathematics class. Under what conditions is the function one-to-one if it assigns to a student his or her

a) mobile phone number.
b) student identification number.
c) final grade in the class.
d) home town.

17. Consider these functions from the set of teachers in a school. Under what conditions is the function one-to-one if it assigns to a teacher his or her

a) office.
b) assigned bus to chaperone in a group of buses taking students on a field trip.
c) salary.
d) social security number.

18. Specify a codomain for each of the functions in Exercise 16. Under what conditions is each of these functions with the codomain you specified onto?

19. Specify a codomain for each of the functions in Exercise 17. Under what conditions is each of the functions with the codomain you specified onto?

20. Give an example of a function from $\mathbf{N}$ to $\mathbf{N}$ that is

a) one-to-one but not onto.
b) onto but not one-to-one.
c) both onto and one-to-one (but different from the identity function).
d) neither one-to-one nor onto.

21. Give an explicit formula for a function from the set of integers to the set of positive integers that is

a) one-to-one, but not onto.
b) onto, but not one-to-one.
c) one-to-one and onto.
d) neither one-to-one nor onto.

22. Determine whether each of these functions is a bijection from $\mathbf{R}$ to $\mathbf{R}$.

a) $f(x) = -3x + 4$
b) $f(x) = -3x^2 + 7$
c) $f(x) = (x + 1)/(x + 2)$
d) $f(x) = x^5 + 1$

23. Determine whether each of these functions is a bijection from $\mathbf{R}$ to $\mathbf{R}$.

a) $f(x) = 2x + 1$
b) $f(x) = x^2 + 1$
c) $f(x) = x^3$
d) $f(x) = (x^2 + 1)/(x^2 + 2)$

24. Let $f: \mathbf{R} \to \mathbf{R}$ and let $f(x) > 0$ for all $x \in \mathbf{R}$. Show that $f(x)$ is strictly increasing if and only if the function $g(x) = 1/f(x)$ is strictly decreasing.

25. Let $f: \mathbf{R} \to \mathbf{R}$ and let $f(x) > 0$ for all $x \in \mathbf{R}$. Show that $f(x)$ is strictly decreasing if and only if the function $g(x) = 1/f(x)$ is strictly increasing.

26. a) Prove that a strictly increasing function from $\mathbf{R}$ to itself is one-to-one.

b) Give an example of an increasing function from $\mathbf{R}$ to itself that is not one-to-one.

27. a) Prove that a strictly decreasing function from $\mathbf{R}$ to itself is one-to-one.

b) Give an example of a decreasing function from $\mathbf{R}$ to itself that is not one-to-one.

28. Show that the function $f(x) = e^x$ from the set of real numbers to the set of real numbers is not invertible, but if the codomain is restricted to the set of positive real numbers, the resulting function is invertible.

29. Show that the function $f(x) = |x|$ from the set of real numbers to the set of nonnegative real numbers is not invertible, but if the domain is restricted to the set of nonnegative real numbers, the resulting function is invertible.

30. Let $S = \{-1, 0, 2, 4, 7\}$. Find $f(S)$ if
   a) $f(x) = 1$.
   b) $f(x) = 2x + 1$.
   c) $f(x) = \lceil x/5 \rceil$.
   d) $f(x) = \lfloor (x^2 + 1)/3 \rfloor$.

31. Let $f(x) = \lfloor x^2/3 \rfloor$. Find $f(S)$ if
   a) $S = \{-2, -1, 0, 1, 2, 3\}$.
   b) $S = \{0, 1, 2, 3, 4, 5\}$.
   c) $S = \{1, 5, 7, 11\}$.
   d) $S = \{2, 6, 10, 14\}$.

32. Let $f(x) = 2x$ where the domain is the set of real numbers. What is
   a) $f(\mathbf{Z})$?
   b) $f(\mathbf{N})$?
   c) $f(\mathbf{R})$?

33. Suppose that g is a function from A to B and f is a function from B to C.
   a) Show that if both f and g are one-to-one functions, then $f \circ g$ is also one-to-one.
   b) Show that if both f and g are onto functions, then $f \circ g$ is also onto.

*34. If f and $f \circ g$ are one-to-one, does it follow that g is one-to-one? Justify your answer.

*35. If f and $f \circ g$ are onto, does it follow that g is onto? Justify your answer.

36. Find $f \circ g$ and $g \circ f$, where $f(x) = x^2 + 1$ and $g(x) = x + 2$, are functions from $\mathbf{R}$ to $\mathbf{R}$.

37. Find $f + g$ and fg for the functions f and g given in Exercise 36.

38. Let $f(x) = ax + b$ and $g(x) = cx + d$, where a, b, c, and d are constants. Determine necessary and sufficient conditions on the constants a, b, c, and d so that $f \circ g = g \circ f$.

39. Show that the function $f(x) = ax + b$ from $\mathbf{R}$ to $\mathbf{R}$ is invertible, where a and b are constants, with $a \neq 0$, and find the inverse of f.

40. Let f be a function from the set A to the set B. Let S and T be subsets of A. Show that
   a) $f(S \cup T) = f(S) \cup f(T)$.
   b) $f(S \cap T) \subseteq f(S) \cap f(T)$.

41. a) Give an example to show that the inclusion in part (b) in Exercise 40 may be proper.
   b) Show that if f is one-to-one, the inclusion in part (b) in Exercise 40 is an equality.

Let f be a function from the set A to the set B. Let S be a subset of B. We define the **inverse image** of S to be the subset of A whose elements are precisely all pre-images of all elements of S. We denote the inverse image of S by $f^{-1}(S)$, so $f^{-1}(S) = \{a \in A \mid f(a) \in S\}$. (*Beware:* The notation f^{-1} is used in two different ways. Do not confuse the notation introduced here with the notation $f^{-1}(y)$ for the value at y of the inverse of the invertible function f. Notice also that $f^{-1}(S)$, the inverse image of the set S, makes sense for all functions f, not just invertible functions.)

42. Let f be the function from $\mathbf{R}$ to $\mathbf{R}$ defined by $f(x) = x^2$. Find
   a) $f^{-1}(\{1\})$.
   b) $f^{-1}(\{x \mid 0 < x < 1\})$.
   c) $f^{-1}(\{x \mid x > 4\})$.

43. Let $g(x) = \lfloor x \rfloor$. Find
   a) $g^{-1}(\{0\})$.
   b) $g^{-1}(\{-1, 0, 1\})$.
   c) $g^{-1}(\{x \mid 0 < x < 1\})$.

44. Let f be a function from A to B. Let S and T be subsets of B. Show that
   a) $f^{-1}(S \cup T) = f^{-1}(S) \cup f^{-1}(T)$.
   b) $f^{-1}(S \cap T) = f^{-1}(S) \cap f^{-1}(T)$.

45. Let f be a function from A to B. Let S be a subset of B. Show that $f^{-1}(\overline{S}) = \overline{f^{-1}(S)}$.

46. Show that $\lfloor x + \frac{1}{2} \rfloor$ is the closest integer to the number x, except when x is midway between two integers, when it is the larger of these two integers.

47. Show that $\lceil x - \frac{1}{2} \rceil$ is the closest integer to the number x, except when x is midway between two integers, when it is the smaller of these two integers.

48. Show that if x is a real number, then $\lceil x \rceil - \lfloor x \rfloor = 1$ if x is not an integer and $\lceil x \rceil - \lfloor x \rfloor = 0$ if x is an integer.

49. Show that if x is a real number, then $x - 1 < \lfloor x \rfloor \leq x \leq \lceil x \rceil < x + 1$.

50. Show that if x is a real number and m is an integer, then $\lceil x + m \rceil = \lceil x \rceil + m$.

51. Show that if x is a real number and n is an integer, then
   a) $x < n$ if and only if $\lfloor x \rfloor < n$.
   b) $n < x$ if and only if $n < \lceil x \rceil$.

52. Show that if x is a real number and n is an integer, then
   a) $x \leq n$ if and only if $\lceil x \rceil \leq n$.
   b) $n \leq x$ if and only if $n \leq \lfloor x \rfloor$.

53. Prove that if n is an integer, then $\lfloor n/2 \rfloor = n/2$ if n is even and $(n - 1)/2$ if n is odd.

54. Prove that if x is a real number, then $\lfloor -x \rfloor = -\lceil x \rceil$ and $\lceil -x \rceil = -\lfloor x \rfloor$.

55. The function INT is found on some calculators, where $\text{INT}(x) = \lfloor x \rfloor$ when x is a nonnegative real number and $\text{INT}(x) = \lceil x \rceil$ when x is a negative real number. Show that this INT function satisfies the identity $\text{INT}(-x) = -\text{INT}(x)$.

56. Let a and b be real numbers with $a < b$. Use the floor and/or ceiling functions to express the number of integers n that satisfy the inequality $a \leq n \leq b$.

57. Let a and b be real numbers with $a < b$. Use the floor and/or ceiling functions to express the number of integers n that satisfy the inequality $a < n < b$.

58. How many bytes are required to encode n bits of data where n equals
   a) 4?
   b) 10?
   c) 500?
   d) 3000?

SOME INFINITE SERIES Although most of the summations in this book are finite sums, infinite series are important in some parts of discrete mathematics. Infinite series are usually studied in a course in calculus and even the definition of these series requires the use of calculus, but sometimes they arise in discrete mathematics, because discrete mathematics deals with infinite collections of discrete elements. In particular, in our future studies in discrete mathematics, we will find the closed forms for the infinite series in Examples 24 and 25 to be quite useful.

EXAMPLE 24 (*Requires calculus*) Let x be a real number with $|x| < 1$. Find $\sum_{n=0}^{\infty} x^n$.

Extra Examples

Solution: By Theorem 1 with $a = 1$ and $r = x$ we see that $\sum_{n=0}^{k} x^n = \frac{x^{k+1} - 1}{x - 1}$. Because $|x| < 1$, x^{k+1} approaches 0 as k approaches infinity. It follows that

$$\sum_{n=0}^{\infty} x^n = \lim_{k\to\infty} \frac{x^{k+1} - 1}{x - 1} = \frac{0 - 1}{x - 1} = \frac{1}{1 - x}.$$

We can produce new summation formulae by differentiating or integrating existing formulae.

EXAMPLE 25 (*Requires calculus*) Differentiating both sides of the equation

$$\sum_{k=0}^{\infty} x^k = \frac{1}{1 - x},$$

from Example 24 we find that

$$\sum_{k=1}^{\infty} kx^{k-1} = \frac{1}{(1 - x)^2}.$$

(This differentiation is valid for $|x| < 1$ by a theorem about infinite series.)

Exercises Section 2.4

1. Find these terms of the sequence $\{a_n\}$, where $a_n = 2 \cdot (-3)^n + 5^n$.
   a) a_0 b) a_1 c) a_4 d) a_5
2. What is the term a_8 of the sequence $\{a_n\}$ if a_n equals
   a) 2^{n-1}? b) 7? c) $1 + (-1)^n$? d) $-(-2)^n$?
3. What are the terms a_0, a_1, a_2, and a_3 of the sequence $\{a_n\}$, where a_n equals
   a) $2^n + 1$? b) $(n + 1)^{n+1}$? c) $\lfloor n/2 \rfloor$? d) $\lfloor n/2 \rfloor + \lceil n/2 \rceil$?
4. What are the terms a_0, a_1, a_2, and a_3 of the sequence $\{a_n\}$, where a_n equals
   a) $(-2)^n$? b) 3? c) $7 + 4^n$? d) $2^n + (-2)^n$?
5. List the first 10 terms of each of these sequences.
   a) the sequence that begins with 2 and in which each successive term is 3 more than the preceding term
   b) the sequence that lists each positive integer three times, in increasing order
   c) the sequence that lists the odd positive integers in increasing order, listing each odd integer twice
   d) the sequence whose nth term is $n! - 2^n$
   e) the sequence that begins with 3, where each succeeding term is twice the preceding term
   f) the sequence whose first term is 2, second term is 4, and each succeeding term is the sum of the two preceding terms
   g) the sequence whose nth term is the number of bits in the binary expansion of the number n (defined in Section 4.2)
   h) the sequence where the nth term is the number of letters in the English word for the index n
6. List the first 10 terms of each of these sequences.
   a) the sequence obtained by starting with 10 and obtaining each term by subtracting 3 from the previous term
   b) the sequence whose nth term is the sum of the first n positive integers
   c) the sequence whose nth term is $3^n - 2^n$
   d) the sequence whose nth term is $\lfloor \sqrt{n} \rfloor$
   e) the sequence whose first two terms are 1 and 5 and each succeeding term is the sum of the two previous terms

f) the sequence whose nth term is the largest integer whose binary expansion (defined in Section 4.2) has n bits (Write your answer in decimal notation.)
g) the sequence whose terms are constructed sequentially as follows: start with 1, then add 1, then multiply by 1, then add 2, then multiply by 2, and so on
h) the sequence whose nth term is the largest integer k such that $k! \leq n$

7. Find at least three different sequences beginning with the terms 1, 2, 4 whose terms are generated by a simple formula or rule.

8. Find at least three different sequences beginning with the terms 3, 5, 7 whose terms are generated by a simple formula or rule.

9. Find the first five terms of the sequence defined by each of these recurrence relations and initial conditions.

a) $a_n = 6a_{n-1}$, $a_0 = 2$
b) $a_n = a_{n-1}^2$, $a_1 = 2$
c) $a_n = a_{n-1} + 3a_{n-2}$, $a_0 = 1$, $a_1 = 2$
d) $a_n = na_{n-1} + n^2 a_{n-2}$, $a_0 = 1$, $a_1 = 1$
e) $a_n = a_{n-1} + a_{n-3}$, $a_0 = 1$, $a_1 = 2$, $a_2 = 0$

10. Find the first six terms of the sequence defined by each of these recurrence relations and initial conditions.

a) $a_n = -2a_{n-1}$, $a_0 = -1$
b) $a_n = a_{n-1} - a_{n-2}$, $a_0 = 2$, $a_1 = -1$
c) $a_n = 3a_{n-1}^2$, $a_0 = 1$
d) $a_n = na_{n-1} + a_{n-2}^2$, $a_0 = -1$, $a_1 = 0$
e) $a_n = a_{n-1} - a_{n-2} + a_{n-3}$, $a_0 = 1$, $a_1 = 1$, $a_2 = 2$

11. Let $a_n = 2^n + 5 \cdot 3^n$ for $n = 0, 1, 2, \ldots$.

a) Find a_0, a_1, a_2, a_3, and a_4.
b) Show that $a_2 = 5a_1 - 6a_0$, $a_3 = 5a_2 - 6a_1$, and $a_4 = 5a_3 - 6a_2$.
c) Show that $a_n = 5a_{n-1} - 6a_{n-2}$ for all integers n with $n \geq 2$.

12. Show that the sequence $\{a_n\}$ is a solution of the recurrence relation $a_n = -3a_{n-1} + 4a_{n-2}$ if

a) $a_n = 0$.
b) $a_n = 1$.
c) $a_n = (-4)^n$.
d) $a_n = 2(-4)^n + 3$.

13. Is the sequence $\{a_n\}$ a solution of the recurrence relation $a_n = 8a_{n-1} - 16a_{n-2}$ if

a) $a_n = 0$?
b) $a_n = 1$?
c) $a_n = 2^n$?
d) $a_n = 4^n$?
e) $a_n = n4^n$?
f) $a_n = 2 \cdot 4^n + 3n4^n$?
g) $a_n = (-4)^n$?
h) $a_n = n^2 4^n$?

14. For each of these sequences find a recurrence relation satisfied by this sequence. (The answers are not unique because there are infinitely many different recurrence relations satisfied by any sequence.)

a) $a_n = 3$
b) $a_n = 2n$
c) $a_n = 2n + 3$
d) $a_n = 5^n$
e) $a_n = n^2$
f) $a_n = n^2 + n$
g) $a_n = n + (-1)^n$
h) $a_n = n!$

15. Show that the sequence $\{a_n\}$ is a solution of the recurrence relation $a_n = a_{n-1} + 2a_{n-2} + 2n - 9$ if

a) $a_n = -n + 2$.
b) $a_n = 5(-1)^n - n + 2$.
c) $a_n = 3(-1)^n + 2^n - n + 2$.
d) $a_n = 7 \cdot 2^n - n + 2$.

16. Find the solution to each of these recurrence relations with the given initial conditions. Use an iterative approach such as that used in Example 10.

a) $a_n = -a_{n-1}$, $a_0 = 5$
b) $a_n = a_{n-1} + 3$, $a_0 = 1$
c) $a_n = a_{n-1} - n$, $a_0 = 4$
d) $a_n = 2a_{n-1} - 3$, $a_0 = -1$
e) $a_n = (n + 1)a_{n-1}$, $a_0 = 2$
f) $a_n = 2na_{n-1}$, $a_0 = 3$
g) $a_n = -a_{n-1} + n - 1$, $a_0 = 7$

17. Find the solution to each of these recurrence relations and initial conditions. Use an iterative approach such as that used in Example 10.

a) $a_n = 3a_{n-1}$, $a_0 = 2$
b) $a_n = a_{n-1} + 2$, $a_0 = 3$
c) $a_n = a_{n-1} + n$, $a_0 = 1$
d) $a_n = a_{n-1} + 2n + 3$, $a_0 = 4$
e) $a_n = 2a_{n-1} - 1$, $a_0 = 1$
f) $a_n = 3a_{n-1} + 1$, $a_0 = 1$
g) $a_n = na_{n-1}$, $a_0 = 5$
h) $a_n = 2na_{n-1}$, $a_0 = 1$

18. A person deposits \$1000 in an account that yields 9% interest compounded annually.

a) Set up a recurrence relation for the amount in the account at the end of n years.
b) Find an explicit formula for the amount in the account at the end of n years.
c) How much money will the account contain after 100 years?

19. Suppose that the number of bacteria in a colony triples every hour.

a) Set up a recurrence relation for the number of bacteria after n hours have elapsed.
b) If 100 bacteria are used to begin a new colony, how many bacteria will be in the colony in 10 hours?

20. Assume that the population of the world in 2010 was 6.9 billion and is growing at the rate of 1.1% a year.

a) Set up a recurrence relation for the population of the world n years after 2010.
b) Find an explicit formula for the population of the world n years after 2010.
c) What will the population of the world be in 2030?

21. A factory makes custom sports cars at an increasing rate. In the first month only one car is made, in the second month two cars are made, and so on, with n cars made in the nth month.

a) Set up a recurrence relation for the number of cars produced in the first n months by this factory.
b) How many cars are produced in the first year?
c) Find an explicit formula for the number of cars produced in the first n months by this factory.

22. An employee joined a company in 2009 with a starting salary of \$50,000. Every year this employee receives a raise of \$1000 plus 5% of the salary of the previous year.

a) Set up a recurrence relation for the salary of this employee n years after 2009.
b) What will the salary of this employee be in 2017?
c) Find an explicit formula for the salary of this employee n years after 2009.

23. Find a recurrence relation for the balance $B(k)$ owed at the end of k months on a loan of \$5000 at a rate of 7% if a payment of \$100 is made each month. [*Hint:* Express $B(k)$ in terms of $B(k-1)$; the monthly interest is $(0.07/12)B(k-1)$.]

24. a) Find a recurrence relation for the balance $B(k)$ owed at the end of k months on a loan at a rate of r if a payment P is made on the loan each month. [*Hint:* Express $B(k)$ in terms of $B(k-1)$ and note that the monthly interest rate is $r/12$.]
b) Determine what the monthly payment P should be so that the loan is paid off after T months.

25. For each of these lists of integers, provide a simple formula or rule that generates the terms of an integer sequence that begins with the given list. Assuming that your formula or rule is correct, determine the next three terms of the sequence.
a) $1, 0, 1, 1, 0, 0, 1, 1, 1, 0, 0, 0, 1, \dots$
b) $1, 2, 2, 3, 4, 4, 5, 6, 6, 7, 8, 8, \dots$
c) $1, 0, 2, 0, 4, 0, 8, 0, 16, 0, \dots$
d) $3, 6, 12, 24, 48, 96, 192, \dots$
e) $15, 8, 1, -6, -13, -20, -27, \dots$
f) $3, 5, 8, 12, 17, 23, 30, 38, 47, \dots$
g) $2, 16, 54, 128, 250, 432, 686, \dots$
h) $2, 3, 7, 25, 121, 721, 5041, 40321, \dots$

26. For each of these lists of integers, provide a simple formula or rule that generates the terms of an integer sequence that begins with the given list. Assuming that your formula or rule is correct, determine the next three terms of the sequence.
a) $3, 6, 11, 18, 27, 38, 51, 66, 83, 102, \dots$
b) $7, 11, 15, 19, 23, 27, 31, 35, 39, 43, \dots$
c) $1, 10, 11, 100, 101, 110, 111, 1000, 1001, 1010, 1011, \dots$
d) $1, 2, 2, 2, 3, 3, 3, 3, 3, 5, 5, 5, 5, 5, 5, 5, \dots$
e) $0, 2, 8, 26, 80, 242, 728, 2186, 6560, 19682, \dots$
f) $1, 3, 15, 105, 945, 10395, 135135, 2027025, 34459425, \dots$
g) $1, 0, 0, 1, 1, 1, 0, 0, 0, 0, 1, 1, 1, 1, 1, \dots$
h) $2, 4, 16, 256, 65536, 4294967296, \dots$

****27.** Show that if a_n denotes the nth positive integer that is not a perfect square, then $a_n = n + \{\sqrt{n}\}$, where $\{x\}$ denotes the integer closest to the real number x.

***28.** Let a_n be the nth term of the sequence $1, 2, 2, 3, 3, 3, 4, 4, 4, 4, 5, 5, 5, 5, 5, 6, 6, 6, 6, 6, 6, \dots$, constructed by including the integer k exactly k times. Show that $a_n = \lfloor \sqrt{2n} + \frac{1}{2} \rfloor$.

29. What are the values of these sums?
a) $\sum_{k=1}^{5} (k+1)$
b) $\sum_{j=0}^{4} (-2)^j$
c) $\sum_{i=1}^{10} 3$
d) $\sum_{j=0}^{8} (2^{j+1} - 2^j)$

30. What are the values of these sums, where $S = \{1, 3, 5, 7\}$?
a) $\sum_{j \in S} j$
b) $\sum_{j \in S} j^2$
c) $\sum_{j \in S} (1/j)$
d) $\sum_{j \in S} 1$

31. What is the value of each of these sums of terms of a geometric progression?
a) $\sum_{j=0}^{8} 3 \cdot 2^j$
b) $\sum_{j=1}^{8} 2^j$
c) $\sum_{j=2}^{8} (-3)^j$
d) $\sum_{j=0}^{8} 2 \cdot (-3)^j$

32. Find the value of each of these sums.
a) $\sum_{j=0}^{8} (1 + (-1)^j)$
b) $\sum_{j=0}^{8} (3^j - 2^j)$
c) $\sum_{j=0}^{8} (2 \cdot 3^j + 3 \cdot 2^j)$
d) $\sum_{j=0}^{8} (2^{j+1} - 2^j)$

33. Compute each of these double sums.
a) $\sum_{i=1}^{2} \sum_{j=1}^{3} (i+j)$
b) $\sum_{i=0}^{2} \sum_{j=0}^{3} (2i+3j)$
c) $\sum_{i=1}^{3} \sum_{j=0}^{2} i$
d) $\sum_{i=0}^{2} \sum_{j=1}^{3} ij$

34. Compute each of these double sums.
a) $\sum_{i=1}^{3} \sum_{j=1}^{2} (i-j)$
b) $\sum_{i=0}^{3} \sum_{j=0}^{2} (3i+2j)$
c) $\sum_{i=1}^{3} \sum_{j=0}^{2} j$
d) $\sum_{i=0}^{2} \sum_{j=0}^{3} i^2 j^3$

35. Show that $\sum_{j=1}^{n} (a_j - a_{j-1}) = a_n - a_0$, where $a_0, a_1, \dots, a_n$ is a sequence of real numbers. This type of sum is called **telescoping**.

36. Use the identity $1/(k(k+1)) = 1/k - 1/(k+1)$ and Exercise 35 to compute $\sum_{k=1}^{n} 1/(k(k+1))$.

37. Sum both sides of the identity $k^2 - (k-1)^2 = 2k - 1$ from $k = 1$ to $k = n$ and use Exercise 35 to find
a) a formula for $\sum_{k=1}^{n} (2k-1)$ (the sum of the first n odd natural numbers).
b) a formula for $\sum_{k=1}^{n} k$.

***38.** Use the technique given in Exercise 35, together with the result of Exercise 37b, to derive the formula for $\sum_{k=1}^{n} k^2$ given in Table 2. [*Hint:* Take $a_k = k^3$ in the telescoping sum in Exercise 35.]

39. Find $\sum_{k=100}^{200} k$. (Use Table 2.)

40. Find $\sum_{k=99}^{200} k^3$. (Use Table 2.)

***41.** Find a formula for $\sum_{k=0}^{m} \lfloor \sqrt{k} \rfloor$, when m is a positive integer.

***42.** Find a formula for $\sum_{k=0}^{m} \lfloor \sqrt[3]{k} \rfloor$, when m is a positive integer.

There is also a special notation for products. The product of $a_m, a_{m+1}, \dots, a_n$ is represented by $\prod_{j=m}^{n} a_j$, read as the product from $j = m$ to $j = n$ of a_j.