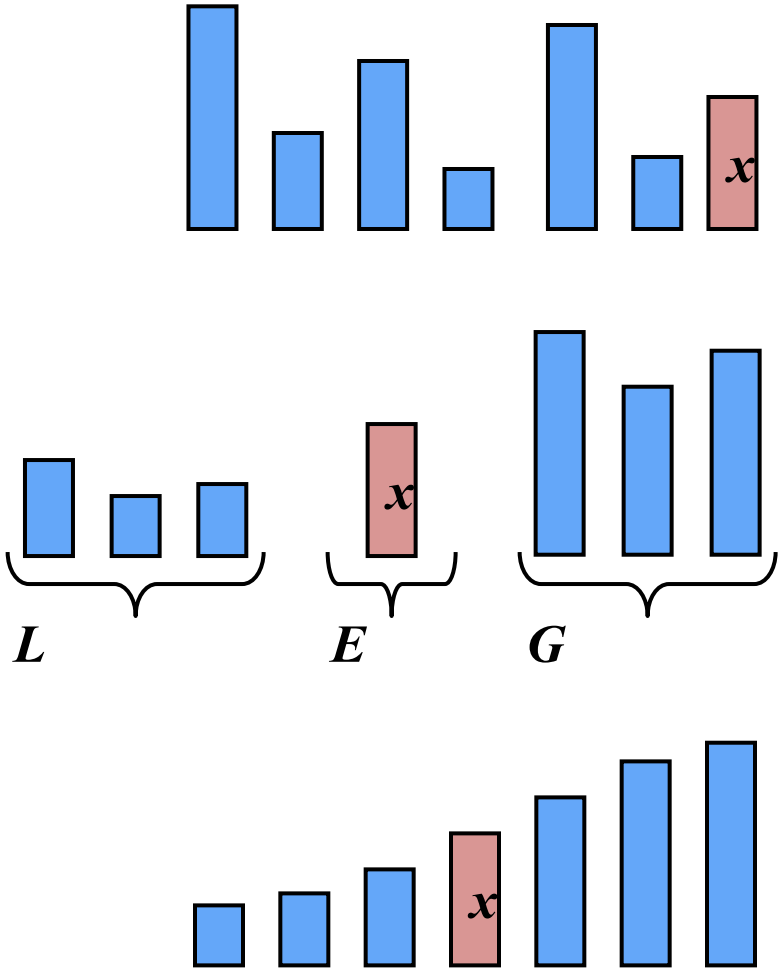
Quick Sort

Quick Sort

A sorting algorithm based on the divide-and-conquer paradigm

- Divide: pick a pivot element x and partition S into
  - L elements less than x
  - E elements equal to x
  - G elements greater than x
- Recur: sort L and G
- Conquer: join L, E and G

The choice of the pivot affects the algorithm's performance.



Partition

1. Remove each element y from S
2. Insert y into L, E or G, depending on the result of the comparison with the pivot x

- Each insert/remove takes $O(1)$ time.
- Thus, the partition step of quick-sort takes $O(n)$ time.


```
Algorithm partition(S, x)
    Input sequence S, pivot element x
    Output subsequences L, E, G
    L, E, G ← empty sequences
    while ¬S.isEmpty()
        y ← S.remove(S.first())
        if y < x
            L.insertLast(y)
        else if y = x
            E.insertLast(y)
        else { y > x }
            G.insertLast(y)
    return L, E, G
```

The choice of the pivot affects the performance of Quick Sort.

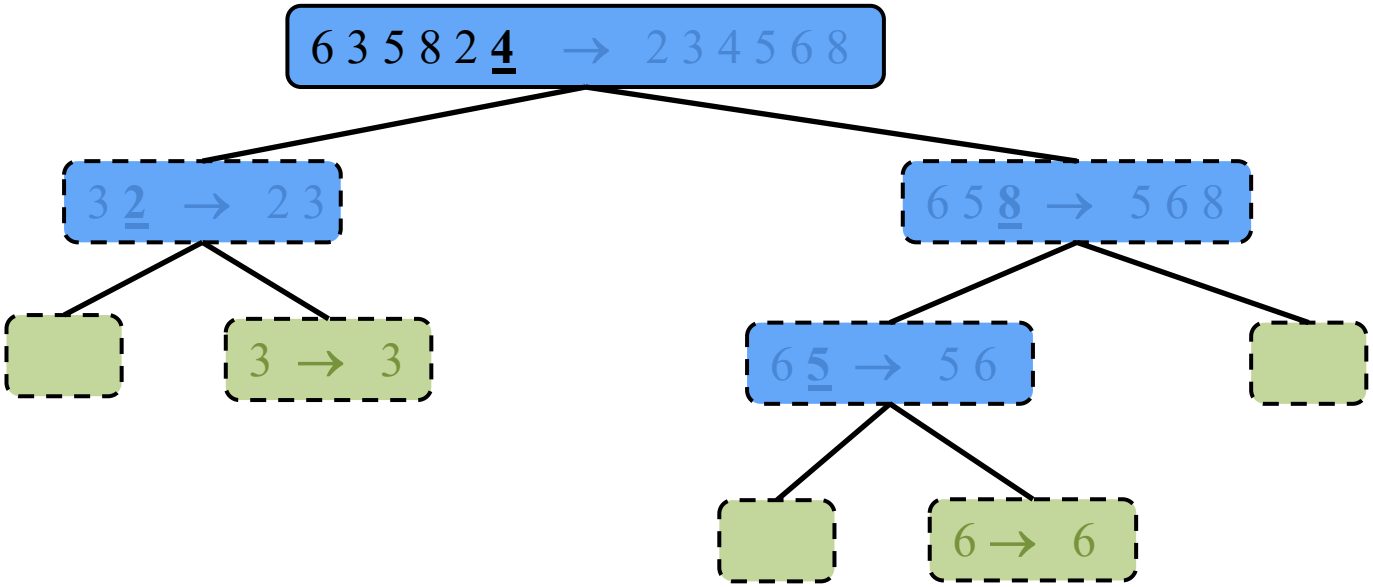
Quick-Sort Tree

An execution of quick-sort depicted by a binary tree

- Each node represents a recursive call of quick-sort and stores
  - Unsorted sequence before the execution and its pivot
  - Sorted sequence at the end of the execution
- The root is the initial call
- The leaves are calls on subsequences of size 0 or 1

Quick Sort Execution

- Strategy: Select the last element as the pivot



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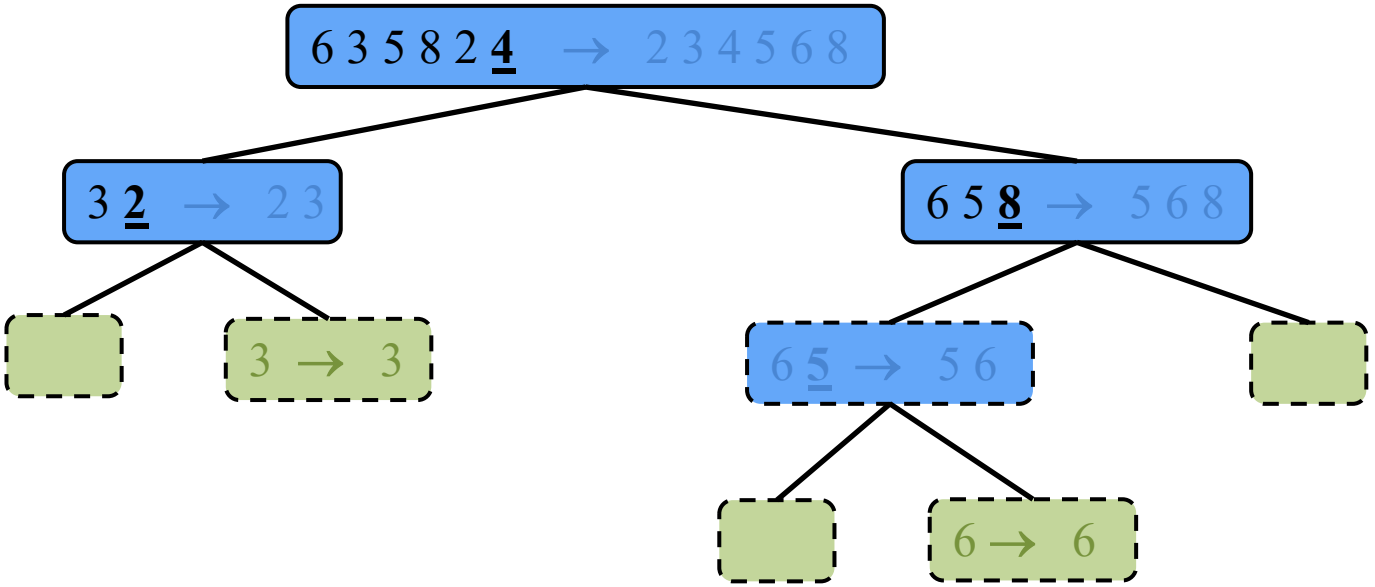

- Select pivot, partition, recursive call

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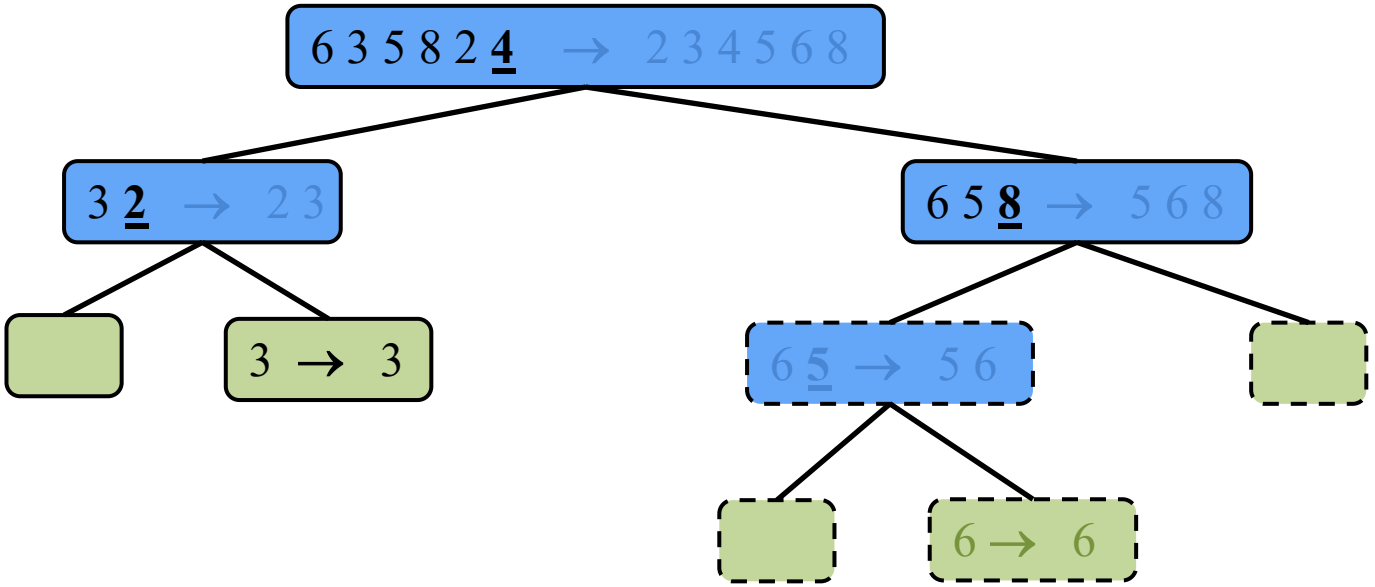

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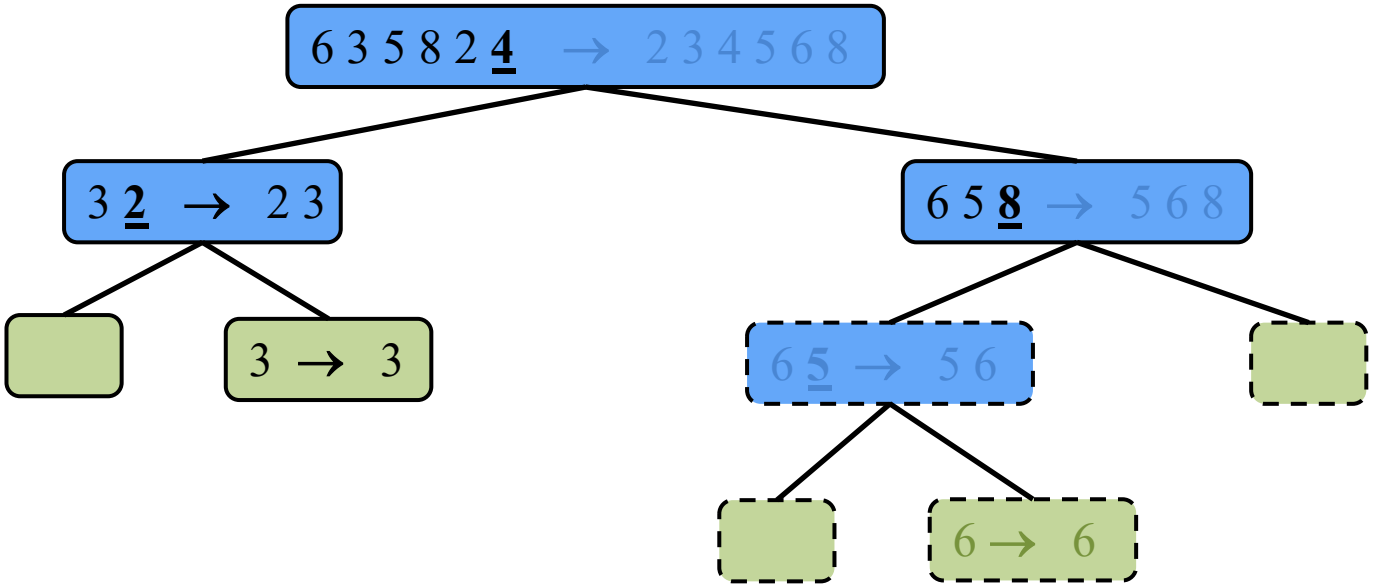
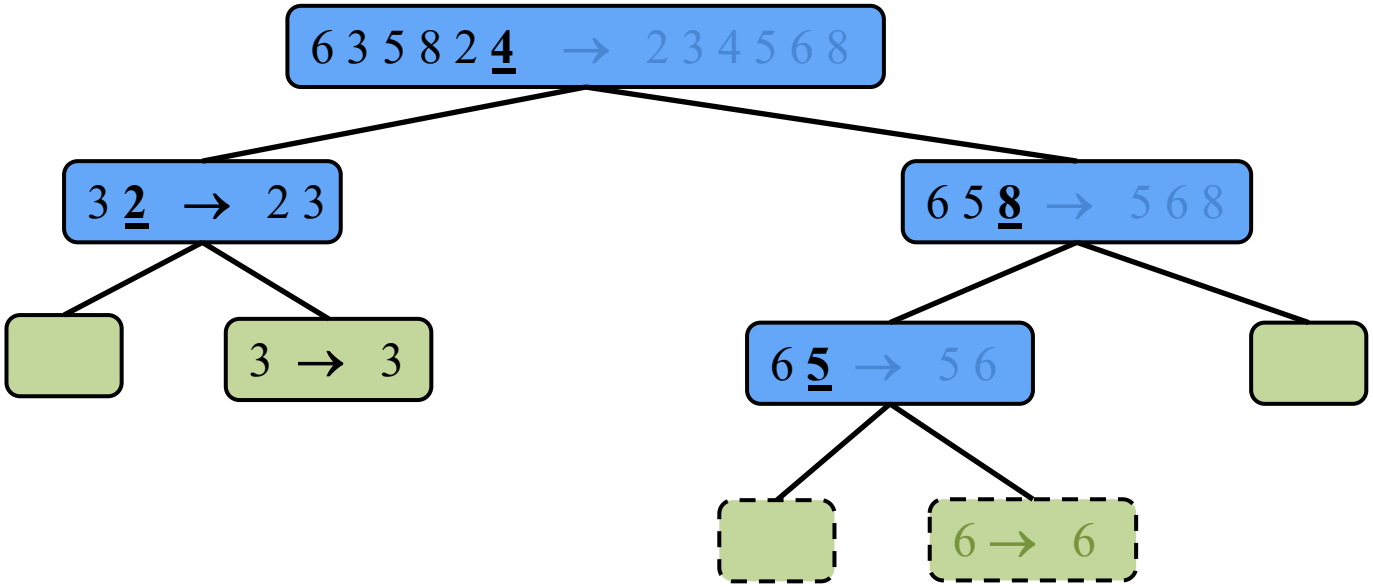


- Join

Quick Sort Execution

- Strategy: Select the last element as the pivot



Quick Sort Execution

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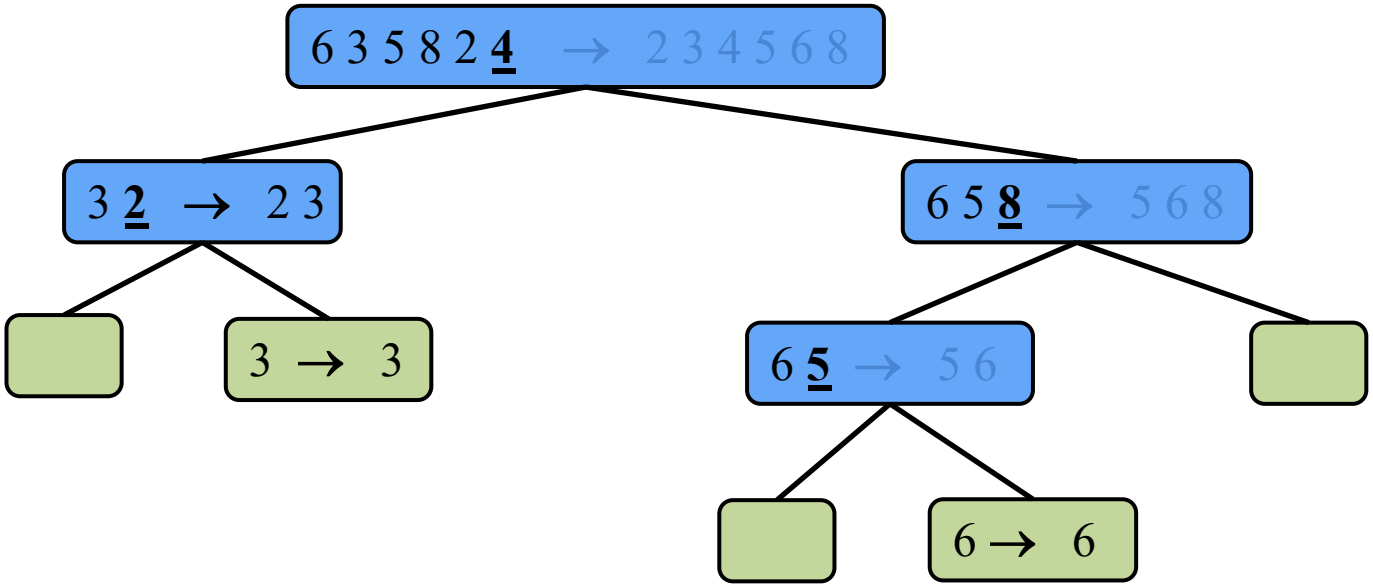

Quick Sort Execution

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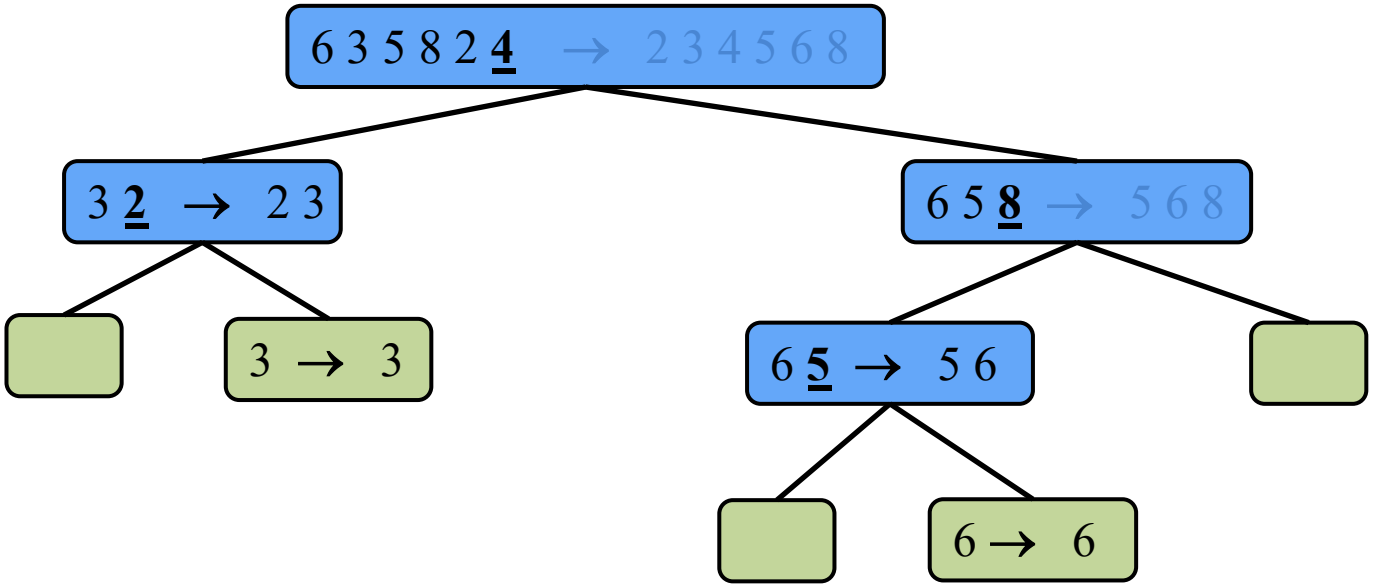

Quick Sort Execution

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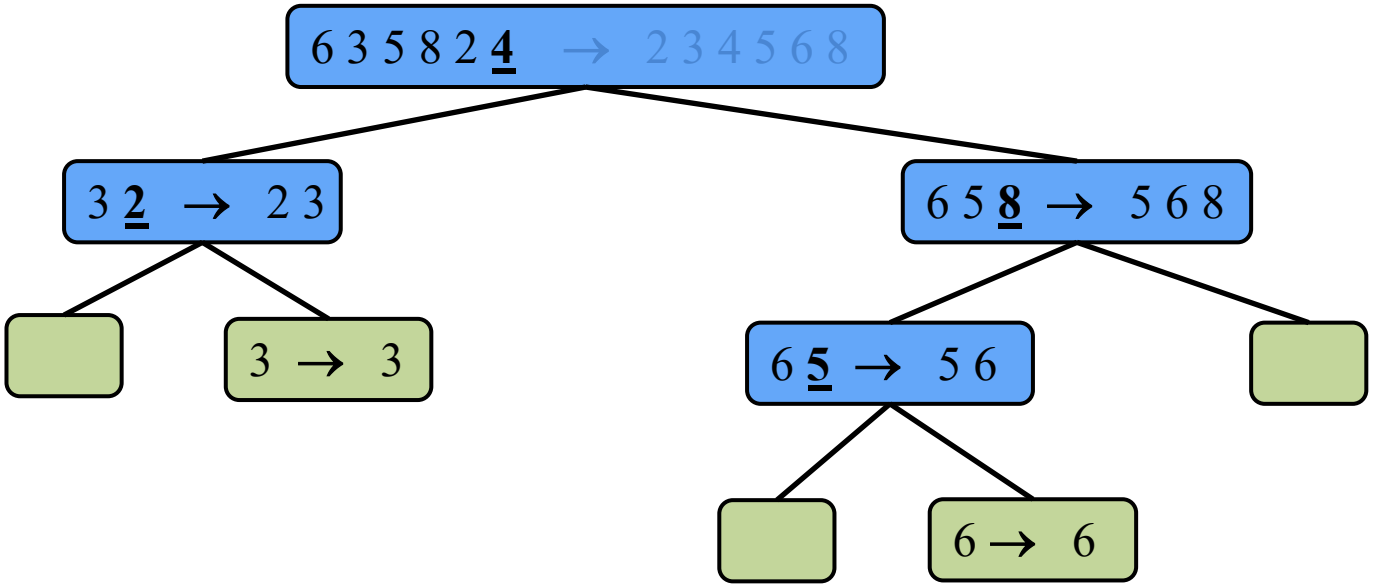
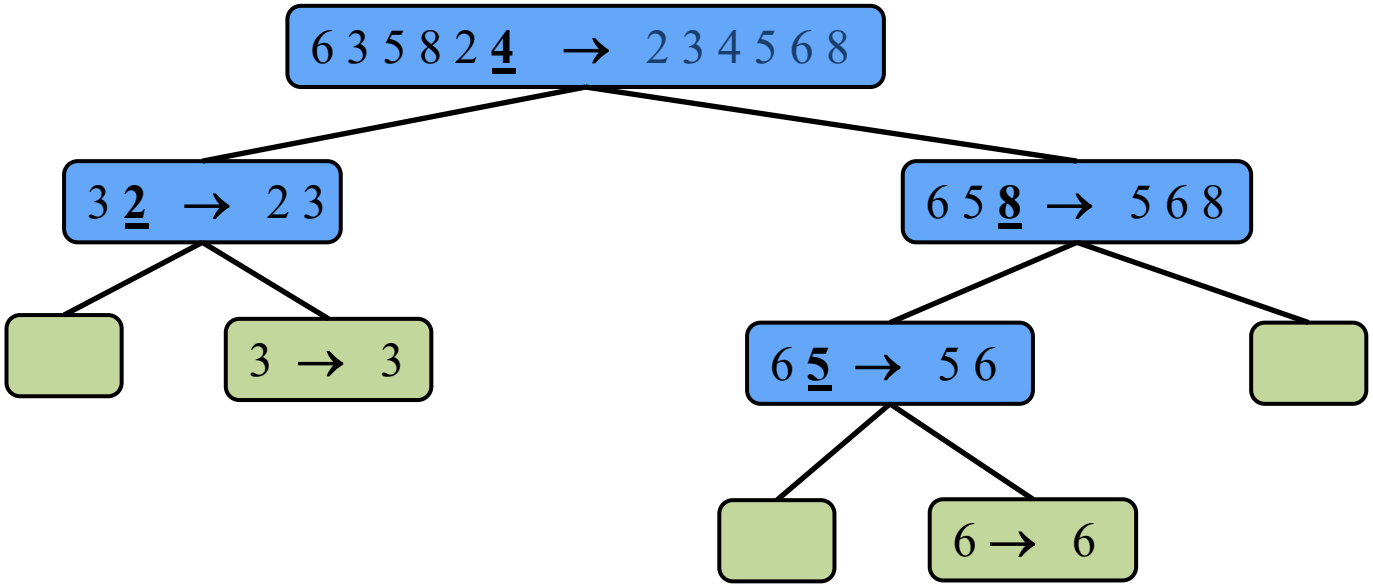


Quick Sort Execution

- Strategy: Select the last element as the pivot

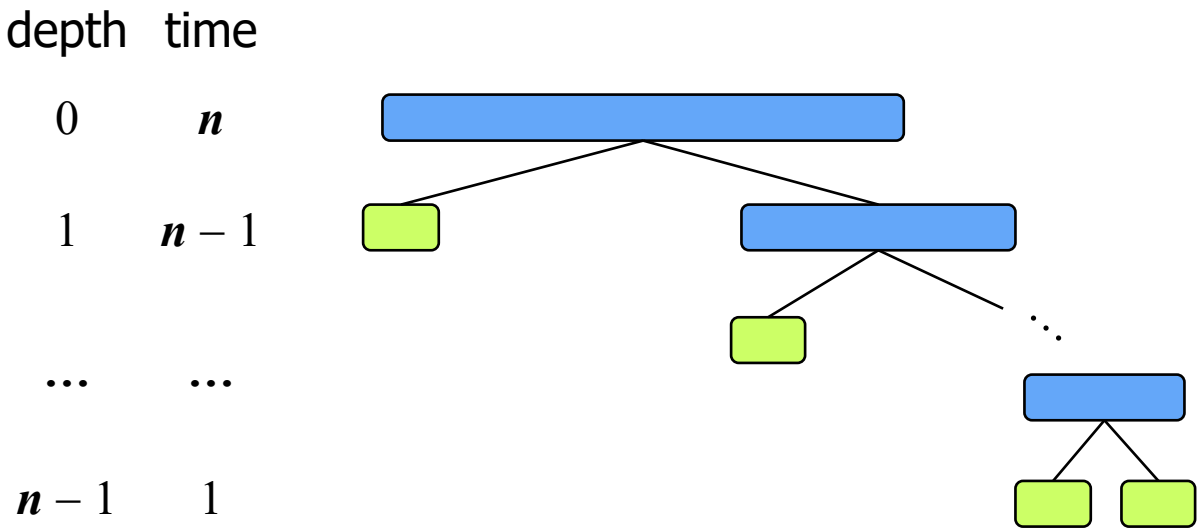


Worst-case Running Time

Occurs when the pivot is the unique minimum or maximum element

- One of L and G has size $n - 1$ and the other has size 0
- The running time is proportional to the sum: $n + (n - 1) + \ldots + 2 + 1$
- If we use the strategy of selecting the **last element** as the pivot, this happens when the list is already sorted!

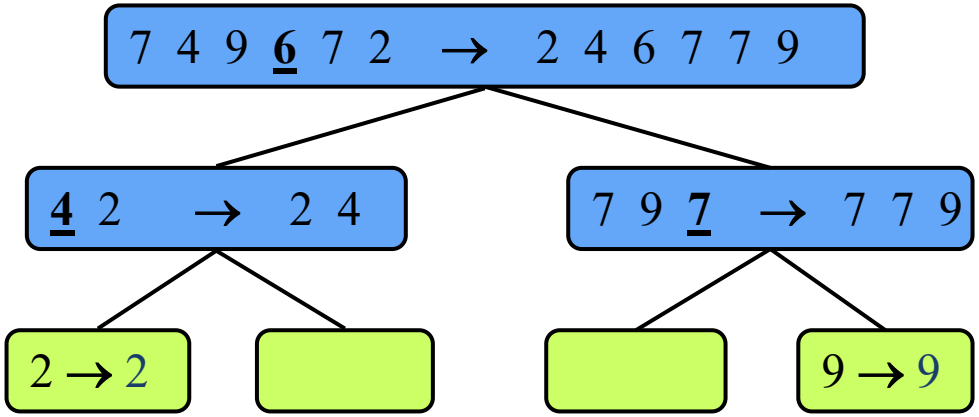
Thus, the worst-case running time of quick-sort is $O(n^2)$



Randomized Quick Sort

Pivot selection strategy: choose a random element as the pivot

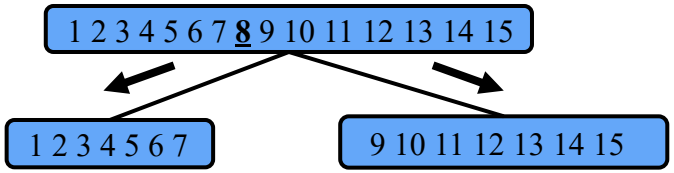
- Still has worst-case running time $O(n^2)$
  - Due to random selection, this case is highly unlikely
- Expected running time is $O(n \log n)$



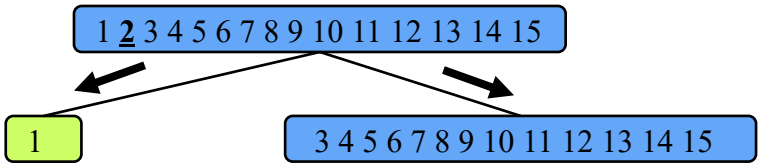
Expected Running Time

Consider a recursive call of quick-sort on a sequence of size s

- **Good call:** the sizes of L and G are each less than $3s/4$
- **Bad call:** one of L and G has size greater than $3s/4$



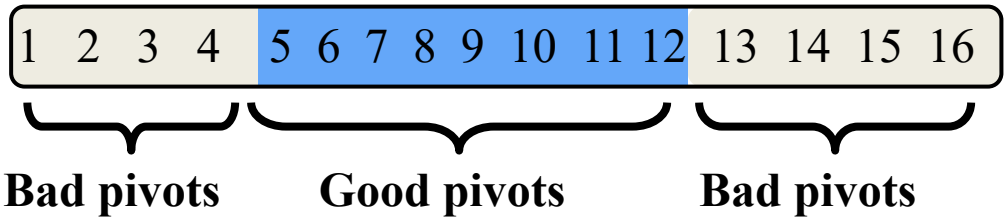
Good call



Bad call

A call is good with probability 1/2

- 1/2 of the possible pivots cause good calls:



Expected Running Time (continued)

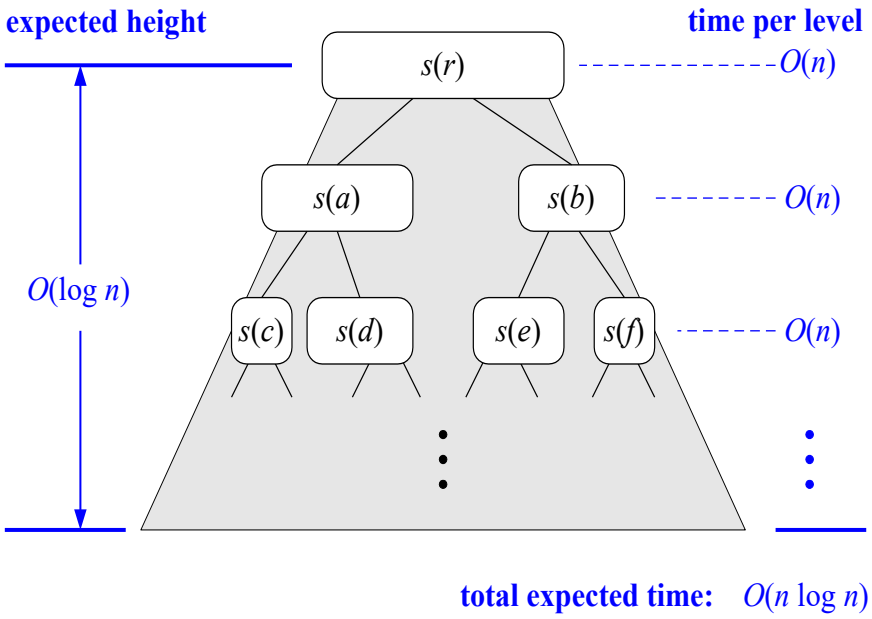
Probabilistic Fact: The expected number of coin tosses required in order to get k heads is $2k$.

For a node of depth i, we expect
- $i/2$ ancestors are good calls
- size of the input sequence for the current call is at most $(3/4)^{i/2}n$

For a node of depth $2\log_{4/3}n$
- the expected input size is one
- the expected height of the quick-sort tree is $O(\log n)$

The amount of work done at the nodes of the same depth is $O(n)$

Thus, the expected running time of quick-sort is $O(n \log n)$



In-Place Quick-Sort

During the partition step, use replace operations to rearrange elements of the input sequences such that:

- elements less than pivot have rank $< h$
- elements equal to pivot have rank between $[h, k]$
- elements greater than pivot have rank $> k$

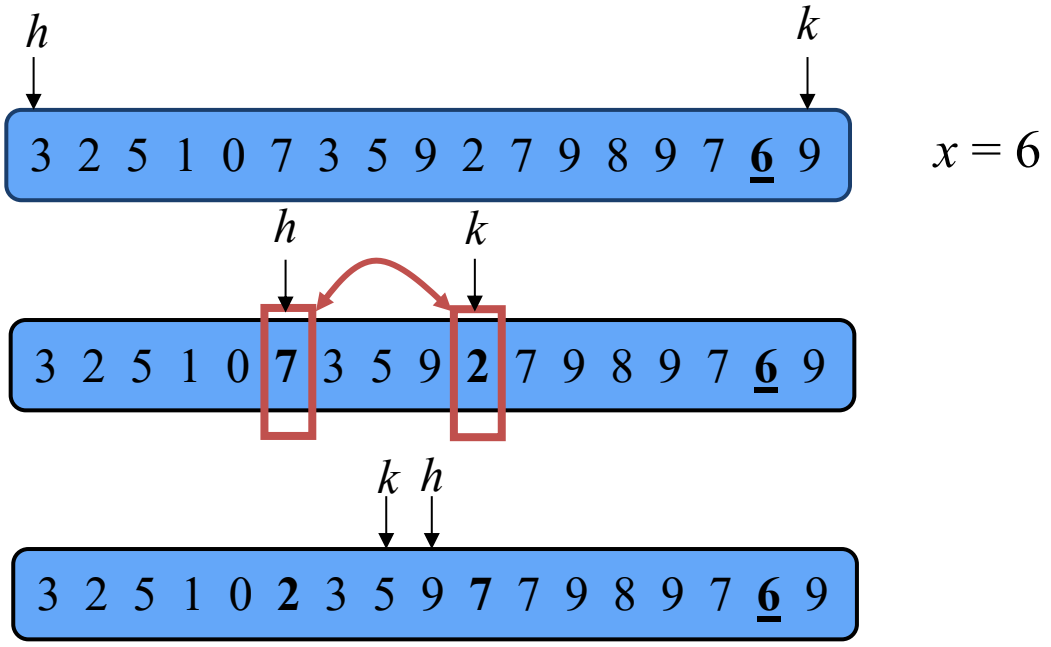
```
Algorithm inPlaceQuickSort(S, l, r)
    Input sequence S, ranks l and r
    Output sequence S with the
        elements of rank between l and r
        rearranged in increasing order
    if l ≥ r
        return
    i ← a random integer between l and r
    x ← S.elemAtRank(i)
    (h, k) ← inPlacePartition(x)
    inPlaceQuickSort(S, l, h − 1)
    inPlaceQuickSort(S, k + 1, r)
```

In-Place Partition

Performs a partitioning using two indices to split S into L and $E \cup G$ (a similar method can split $E \cup G$ into E and G).

Repeat until h and k cross:

- Scan h to the right until it finds an element $\geq x$
- Scan k to the left until it finds an element $< x$
- Swap elements at indices h and k



Summary of Sorting Algorithms

Algorithm	Time	Notes
selection-sort	$O(n^2)$	◆ in-place, not stable ◆ slow (good for small inputs)
insertion-sort	$O(n^2)$	◆ in-place, stable ◆ slow (good for small inputs)
quick-sort	$O(n \log n)$ expected	◆ in-place, not stable ◆ randomized ◆ fast (good for large inputs)
heap-sort	$O(n \log n)$	◆ in-place, not stable ◆ fast (good for large inputs)
merge-sort	$O(n \log n)$	◆ not in-place, stable ◆ sequential data access ◆ fast (good for huge inputs)

Other: Nuts and Bolts

You are given a collection of *n* bolts of different widths, and *n* corresponding nuts.

- You can test whether a given nut and bolt fit together, from which you learn whether the nut is too large, too small, or an exact match for the bolt.
- The differences in size between pairs of nuts or bolts are too small to see by eye, so you cannot compare the sizes of two nuts or two bolts directly.
- You are to match each bolt to each nut.

Give an efficient algorithm to solve the nuts and bolts problem.