Divide-and-Conquer

Divide-and-conquer is a general algorithm design paradigm:

- Divide: divide the input data S in two disjoint subsets S_1 and S_2
- Recur: solve the subproblems associated with S_1 and S_2
 - the base case for the recursion are subproblems of size 0 or 1
- Conquer: combine the solutions for S_1 and S_2 into a solution for S

Merge-sort is a sorting algorithm based on the divide-and-conquer paradigm

- Like heap-sort
 - Uses a comparator
 - Has $O(n \log n)$ running time
- Unlike heap-sort
 - Does not use an auxiliary priority queue
 - Accesses data in a sequential manner (suitable to sort data on a disk)

Merge Sort

Merge-sort on an input sequence S with n elements consists of three steps:

- Divide: partition S into two sequences S_1 and S_2 of about n/2 elements each
- Recur: recursively sort S_1 and S_2
- Conquer: merge S_1 and S_2 into a unique sorted sequence

```
Algorithm mergeSort(S, C)
Input sequence S with n elements, comparator C
Output sequence S sorted according to C
if S.size() > 1
(S_1, S_2) \leftarrow partition(S, n/2)
mergeSort(S_1, C)
mergeSort(S_2, C)
S \leftarrow merge(S_1, S_2)
```

Merging two sorted sequences

The conquer step of mergesort consists of merging two sorted sequences A and B into a sorted sequence Scontaining the union of the elements of A and B

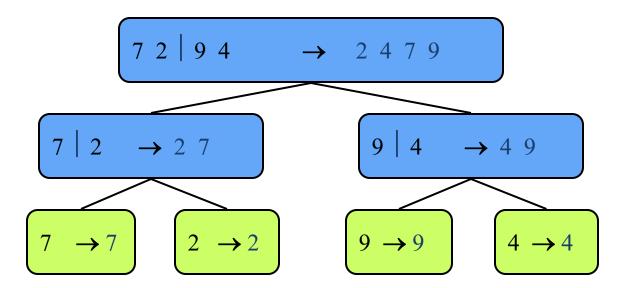
Merging two sorted sequences, each with n/2 elements, takes O(n) time

```
Algorithm merge(A, B)
    Input sequences \boldsymbol{A} and \boldsymbol{B} with \boldsymbol{n}/2 elements each
    Output sorted sequence of A \cup B
    S \leftarrow empty sequence
    while \neg A.isEmpty() \land \neg B.isEmpty()
        if A.first().element() \leq B.first().element()
            S.insertLast(A.remove(A.first()))
        else
            S.insertLast(B.remove(B.first()))
    while \neg A.isEmpty()
        S.insertLast(A.remove(A.first()))
    while \neg B.isEmpty()
        S.insertLast(B.remove(B.first()))
    return S
```

Merge-Sort Tree

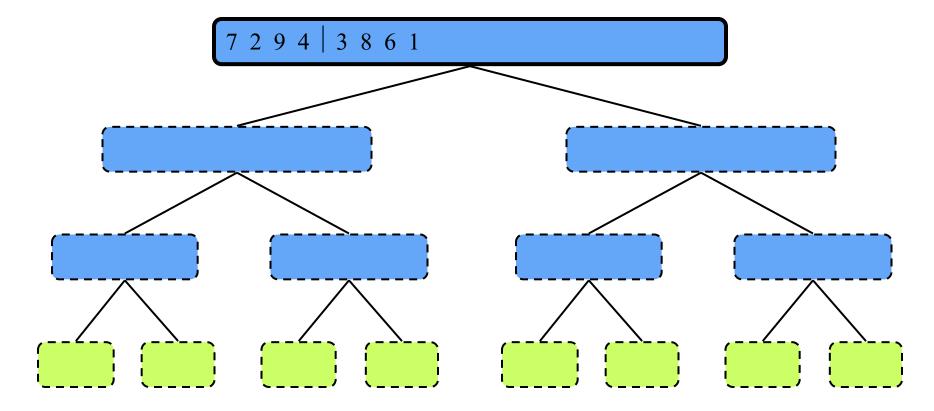
An execution of merge-sort is depicted by a binary tree

- each node represents a recursive call of merge-sort and stores
 - unsorted sequence before the execution and its partition
 - sorted sequence at the end of the execution
- the root is the initial call
- the leaves are calls on subsequences of size 1

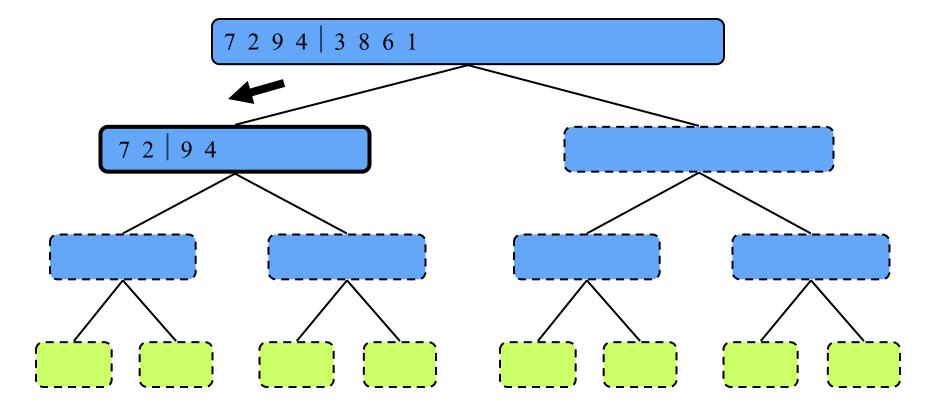


Execution Example

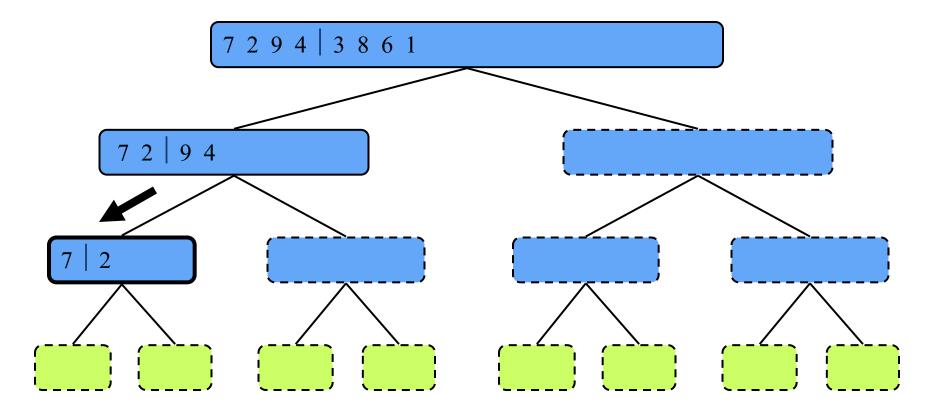
• Partition



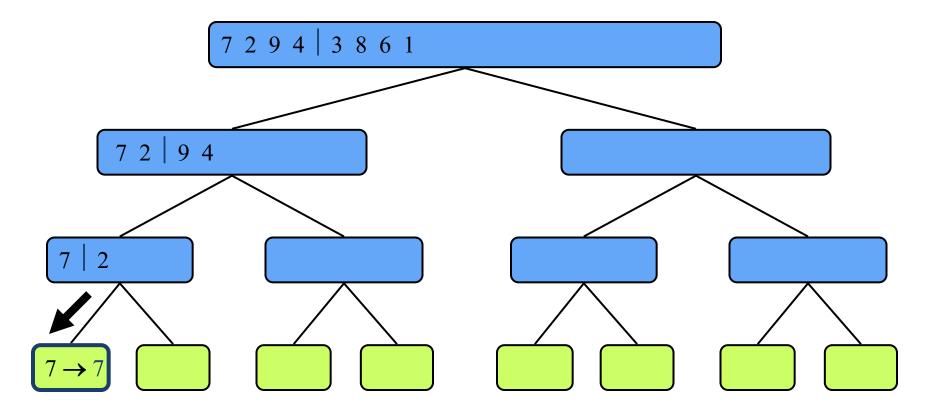
• Recursive call, partition



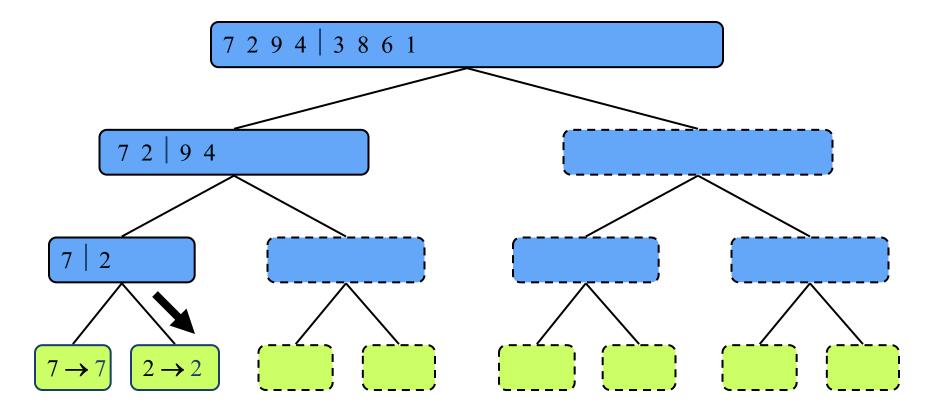
• Recursive call, partition



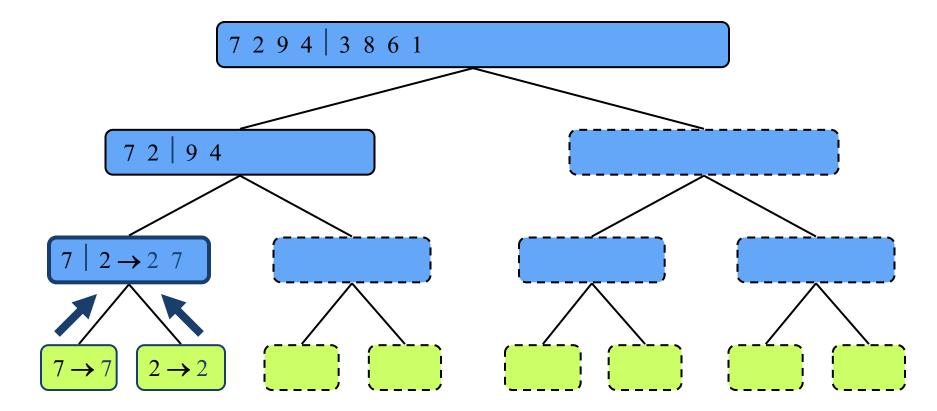
• Recursive call, base case



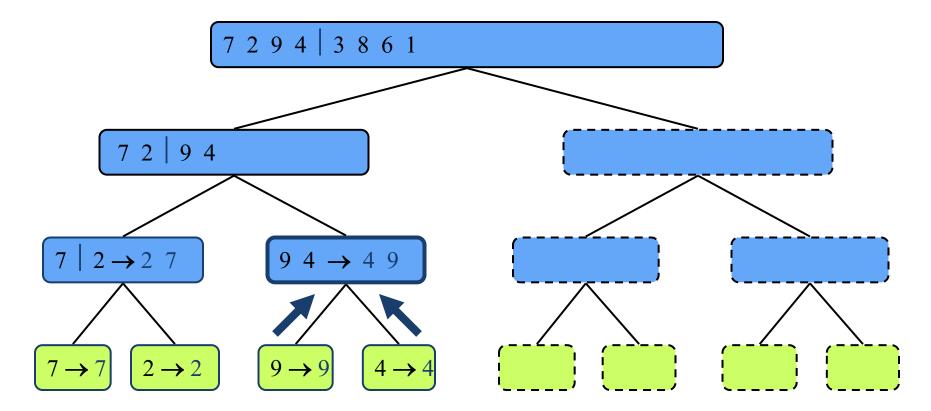
• Recursive call, base case



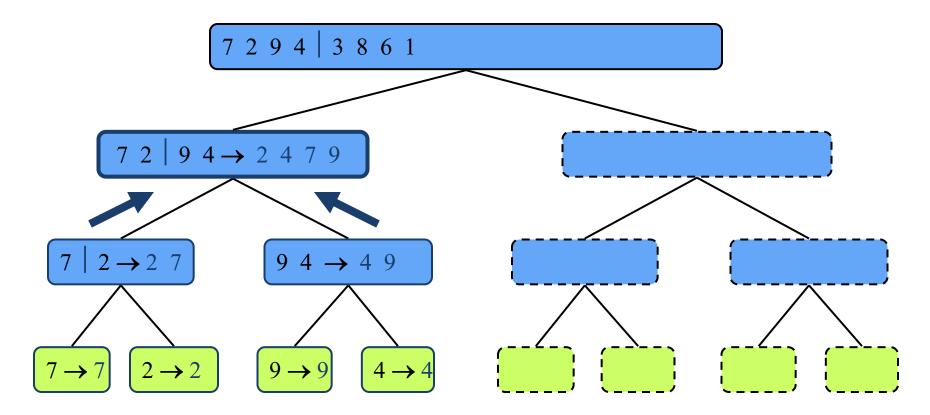
Merge

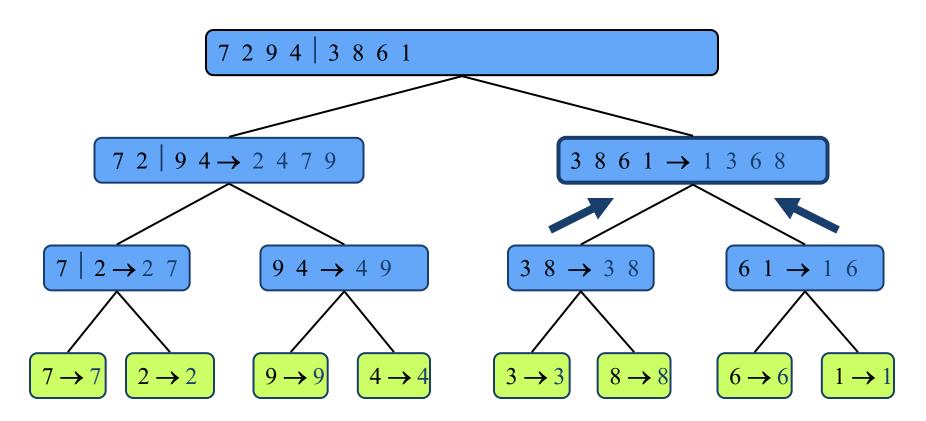


• Recursive call, ..., base case, merge

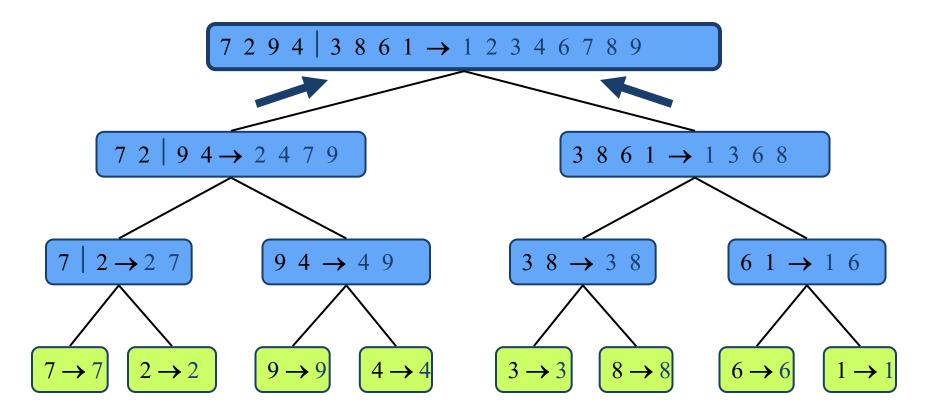


Merge





Merge



Analysis of Merge-Sort

- The height h of the merge-sort tree is $O(\log n)$
 - at each recursive call we divide the sequence in half
- The overall amount or work done at the nodes of depth i is O(n)
 - we partition and merge 2^i sequences of size $n/2^i$
 - we make 2^{i+1} recursive calls
- Thus, the total running time of merge-sort is $O(n \log n)$

Comparing sorting algorithms

Consider the following when evaluating a sorting algorithm:

- Time complexity
- Space complexity
 - An in-place algorithm requires only n + O(1) space, using the already given space for the n elements and O(1) additional space
- Stability
 - A sorting algorithm is stable if it preserves the original relative ordering of elements with equal value
 - Ex: Unsorted sequence (\mathbf{B} , \mathbf{b} , \mathbf{a} , \mathbf{c}). Suppose $\mathbf{B} = \mathbf{b}$ and $\mathbf{a} < \mathbf{b} < \mathbf{c}$.
 - Stable sorted: (a, **B**, b, c)
 - Unstable sorted: (a, b, B, c)
 - Necessary if we want to sort repeatedly by different attributes
 (i.e., sort by first name, then sort again by last name)

Summary of Sorting Algorithms

Algorithm	Time	Notes
selection-sort	$O(n^2)$	in-placenot stablefor small data sets (< 1K)
insertion-sort	$O(n^2)$	in-placestablefor small data sets (< 1K)
heap-sort	$O(n \log n)$	 in-place not stable for large data sets (1K — 1M)
merge-sort	$O(n \log n)$	 not in-place stable sequential data access for huge data sets (> 1M)

Other

• You are given a query point p and a set S of n other points in two dimensional space. Find k points out of the n points which are nearest to p.

