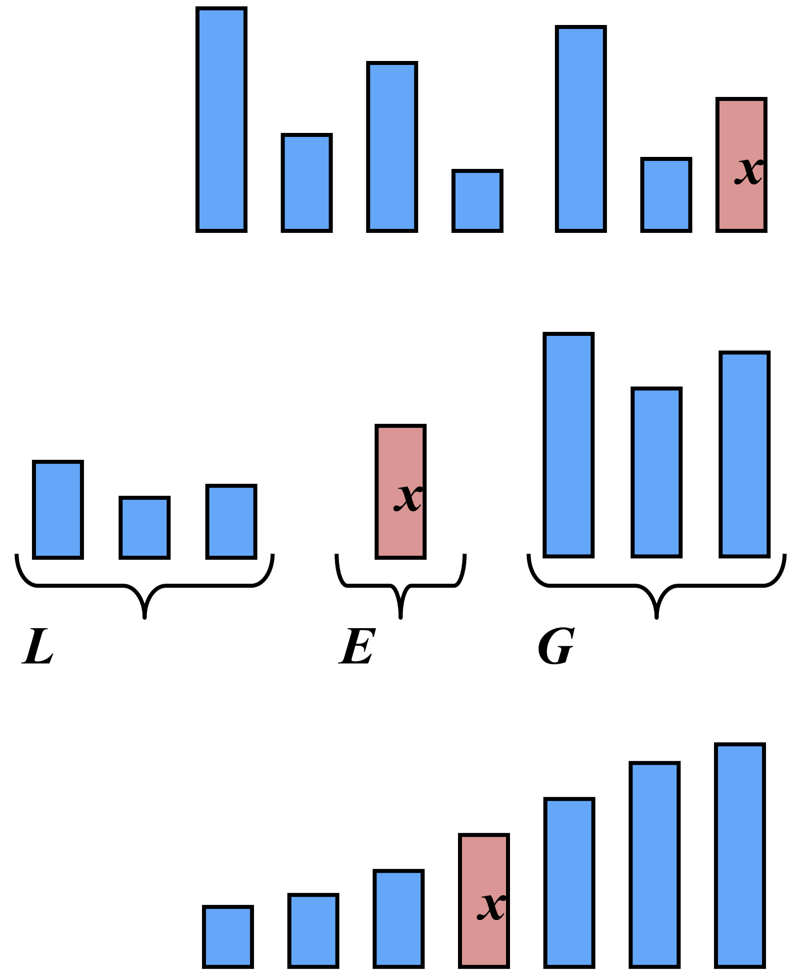
# Quick Sort

# Quick Sort

A sorting algorithm based on the divide-and-conquer paradigm

- Divide: pick a pivot element $x$ and partition $S$ into
  - $L$ elements less than $x$
  - $E$ elements equal to $x$
  - $G$ elements greater than $x$
- Recur: sort $L$ and $G$
- Conquer: join $L$, $E$ and $G$

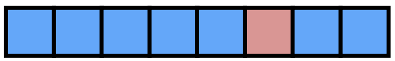
The choice of the pivot affects the algorithm's performance.



# Partition

1. Remove each element $y$ from $S$
2. Insert $y$ into $L$, $E$ or $G$, depending on the result of the comparison with the pivot $x$

- Each insert/remove takes $O(1)$ time.
- Thus, the partition step of quick-sort takes $O(n)$ time.




**Algorithm** *partition*($S$, $x$)
    **Input** sequence $S$, pivot element $x$
    **Output** subsequences $L, E, G$
    $L, E, G \leftarrow$ empty sequences
    **while** $\neg S.isEmpty()$
        $y \leftarrow S.remove(S.first())$
        **if** $y < x$
            $L.insertLast(y)$
        **else if** $y = x$
            $E.insertLast(y)$
        **else** { $y > x$ }
            $G.insertLast(y)$
    **return** $L, E, G$

The choice of the pivot affects the performance of Quick Sort.

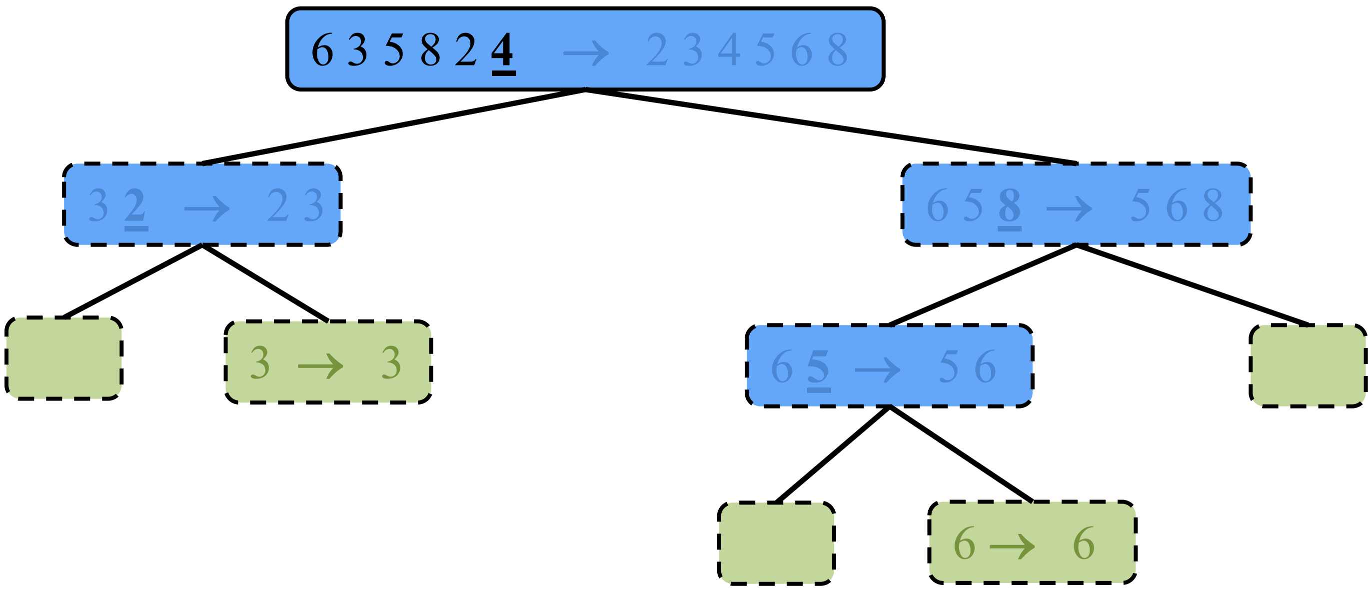
# Quick-Sort Tree

An execution of quick-sort depicted by a binary tree

- Each node represents a recursive call of quick-sort and stores
  - Unsorted sequence before the execution and its pivot
  - Sorted sequence at the end of the execution
- The root is the initial call
- The leaves are calls on subsequences of size 0 or 1

# Quick Sort Execution

- Strategy: Select the last element as the pivot



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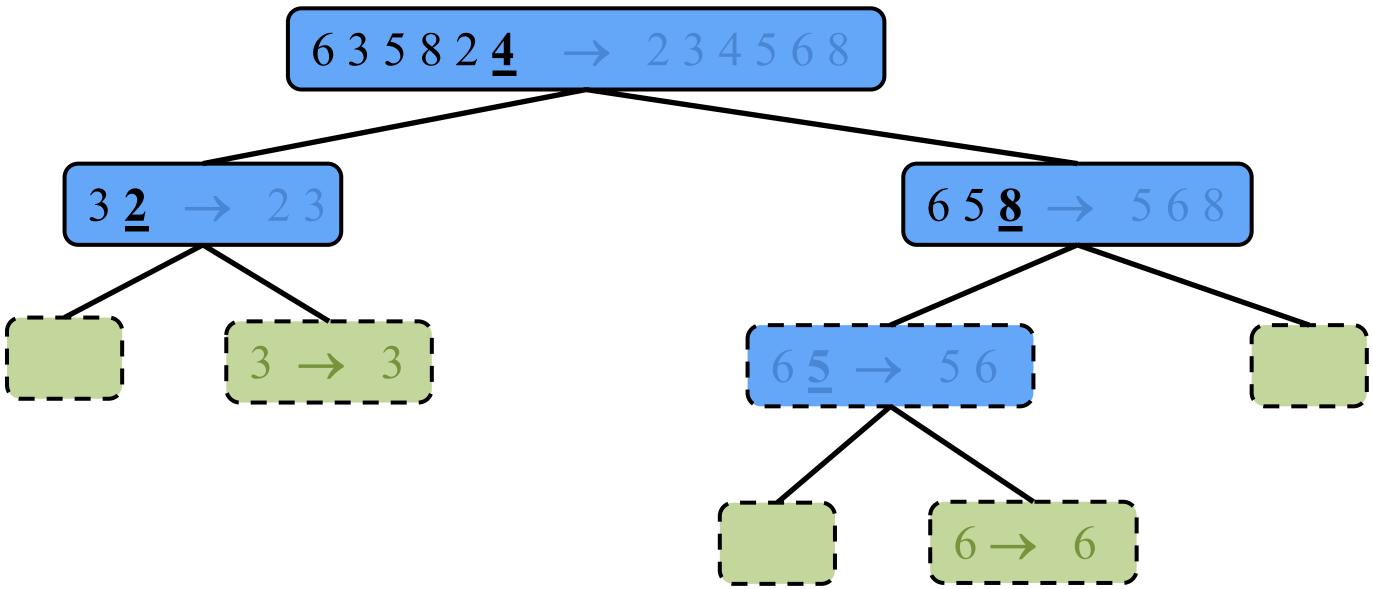


- Select pivot, partition, recursive call

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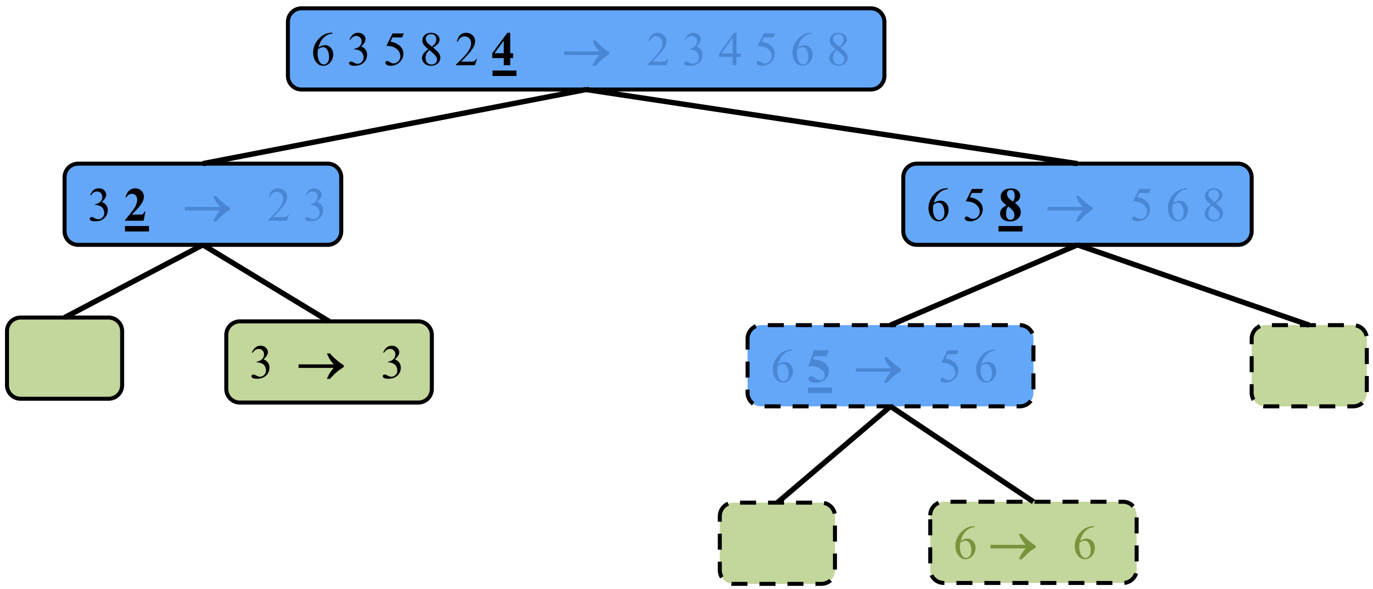


- Select pivot, partition, recursive call

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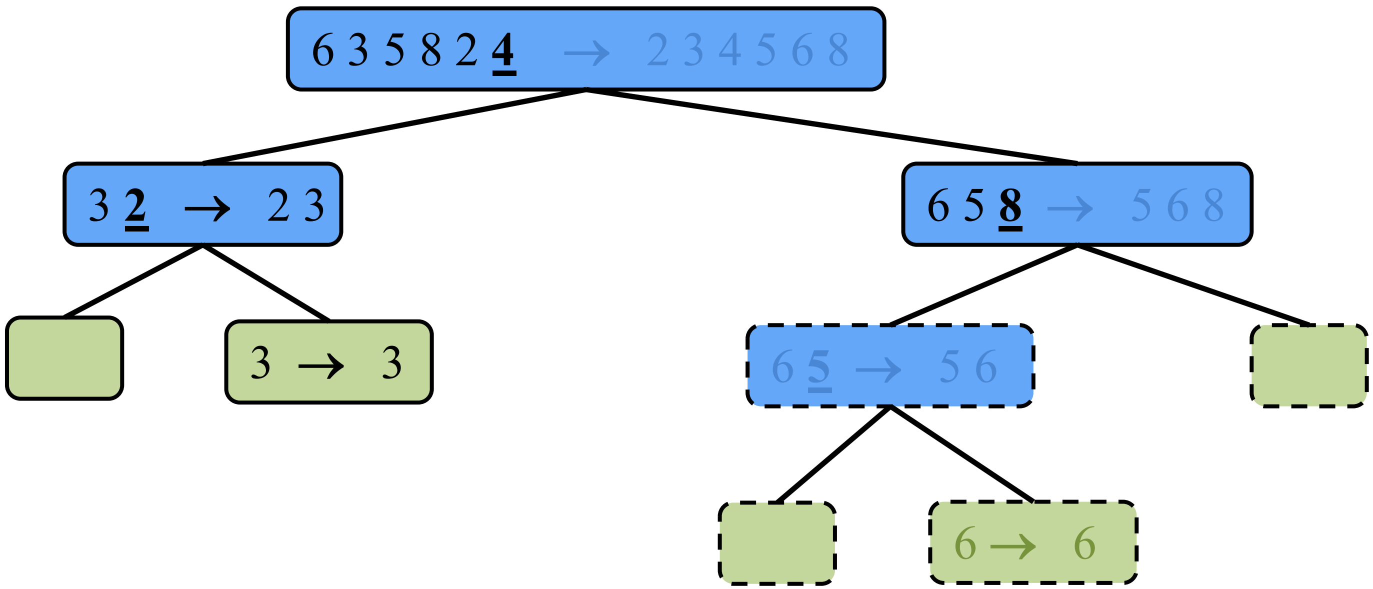
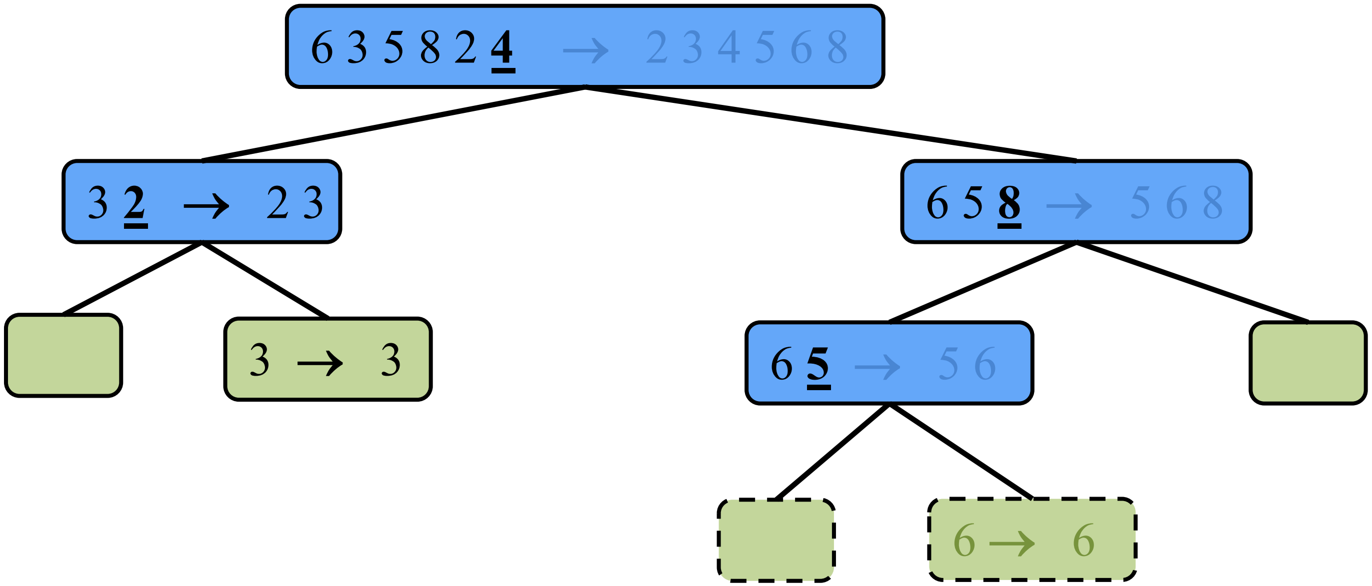



- Join

# Quick Sort Execution

- Strategy: Select the last element as the pivot



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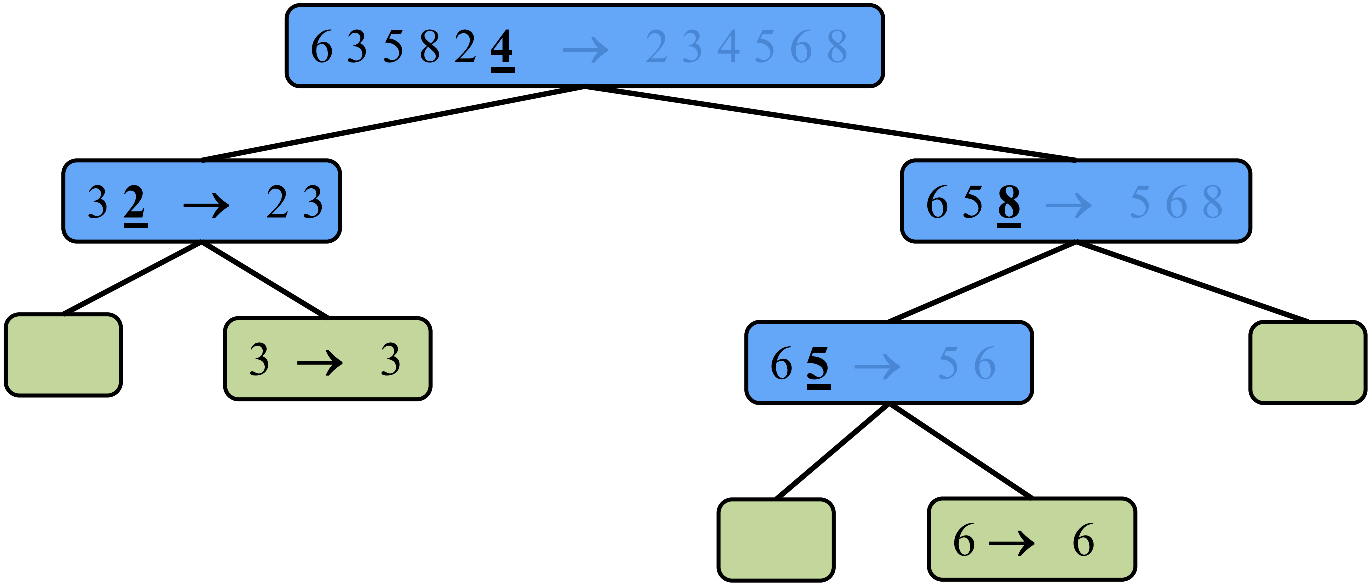

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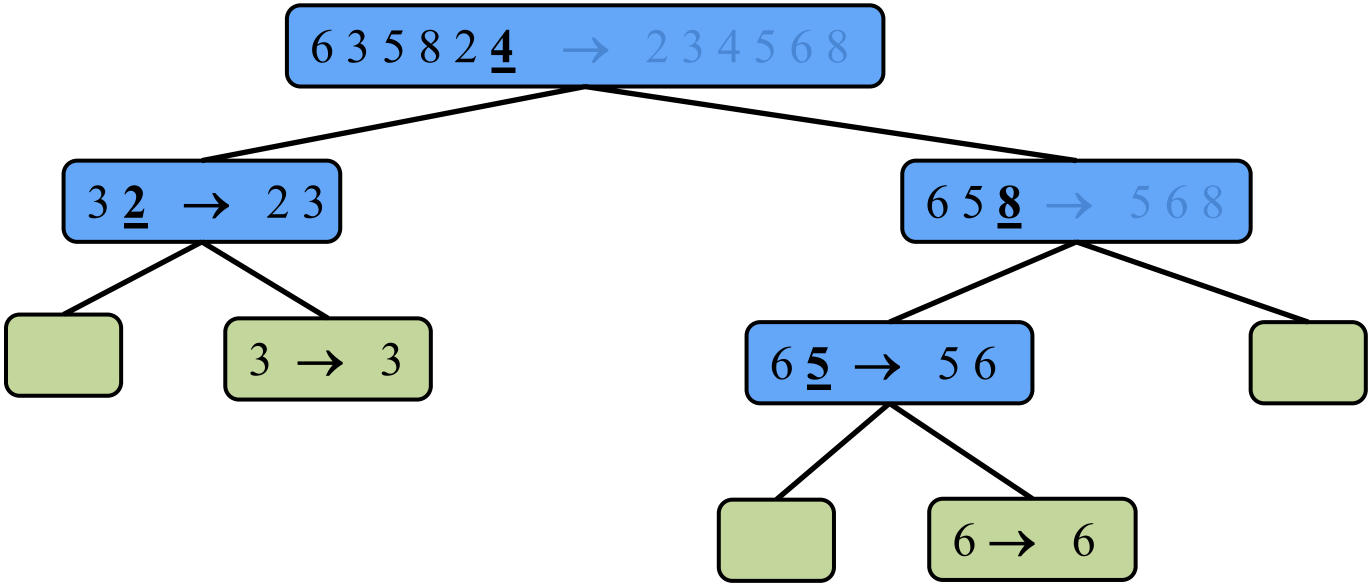

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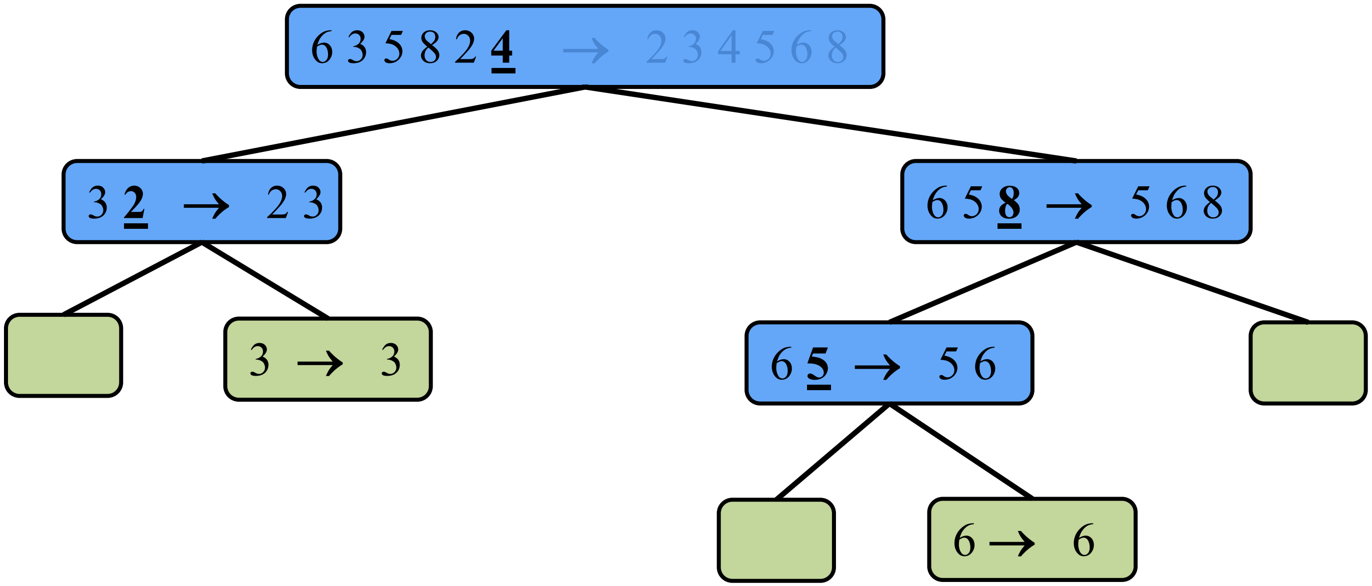

# Quick Sort Execution

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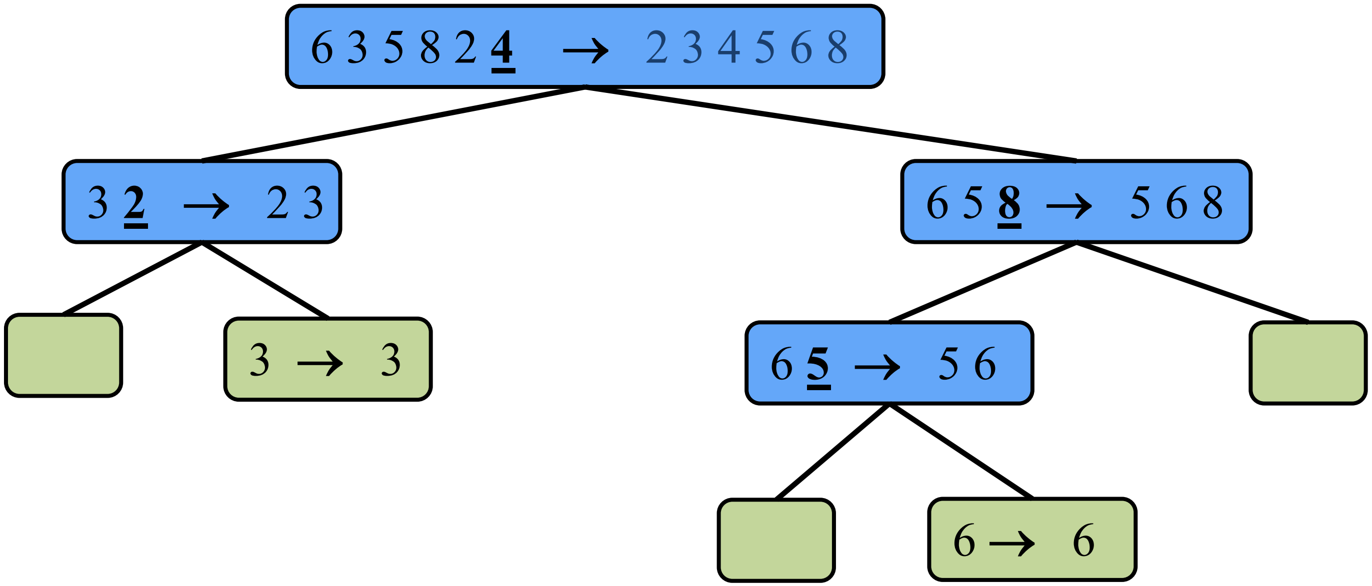


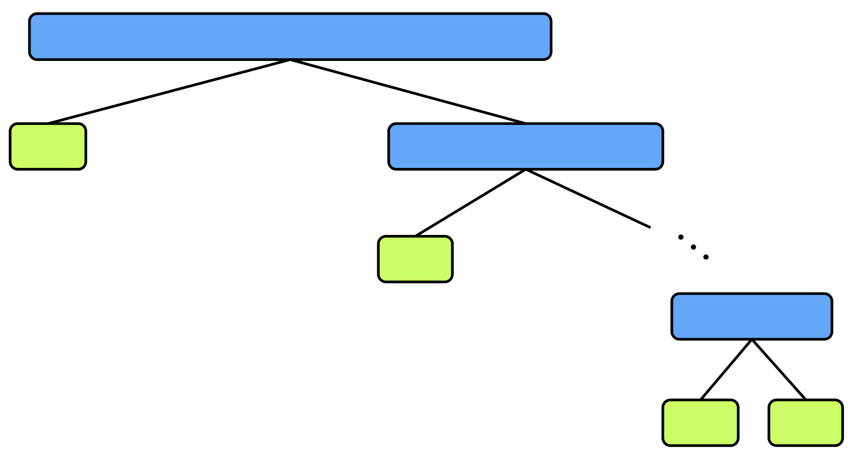
# Worst-case Running Time

Occurs when the pivot is the unique minimum or maximum element

- One of $\boldsymbol{L}$ and $\boldsymbol{G}$ has size $\boldsymbol{n} - 1$ and the other has size 0
- The running time is proportional to the sum: $\boldsymbol{n} + (\boldsymbol{n} - 1) + \ldots + 2 + 1$
- If we use the strategy of selecting the **last element** as the pivot, this happens when the list is already sorted!

Thus, the worst-case running time of quick-sort is $\boldsymbol{O}(\boldsymbol{n}^2)$

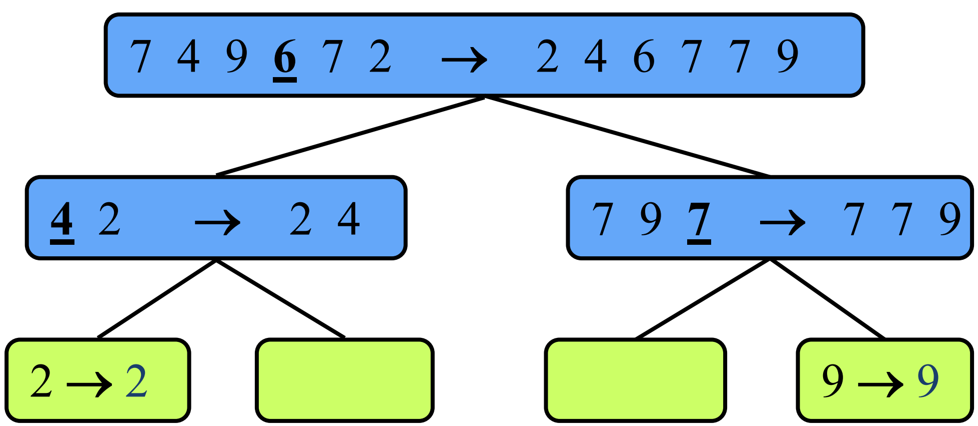
| depth | time |
|---|---|
| 0 | $\boldsymbol{n}$ |
| 1 | $\boldsymbol{n} - 1$ |
| … | … |
| $\boldsymbol{n} - 1$ | 1 |



# Randomized Quick Sort

Pivot selection strategy: choose a random element as the pivot

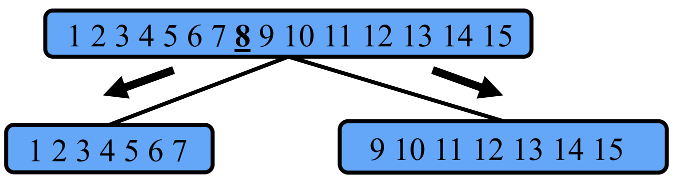
- Still has worst-case running time $O(n^2)$
  - Due to random selection, this case is highly unlikely
- Expected running time is $O(n \log n)$



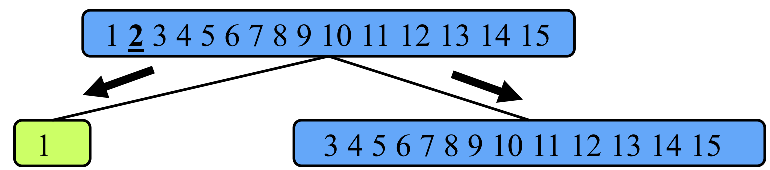
# Expected Running Time

Consider a recursive call of quick-sort on a sequence of size $s$

- **Good call:** the sizes of $L$ and $G$ are each less than $3s/4$
- **Bad call:** one of $L$ and $G$ has size greater than $3s/4$



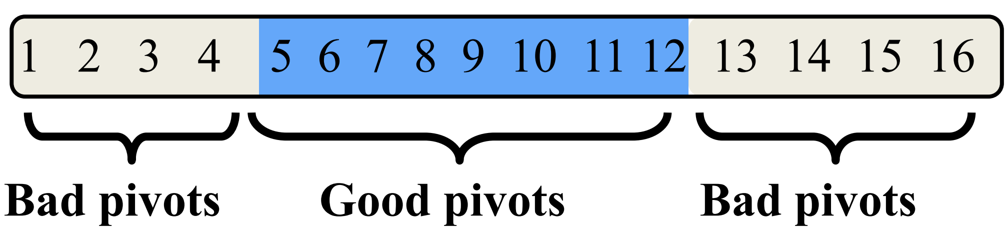

**Good call**




**Bad call**

A call is good with probability 1/2

- 1/2 of the possible pivots cause good calls:



# Expected Running Time (continued)

Probabilistic Fact: The expected number of coin tosses required in order to get $\boldsymbol{k}$ heads is $2\boldsymbol{k}$.

For a node of depth $i$, we expect

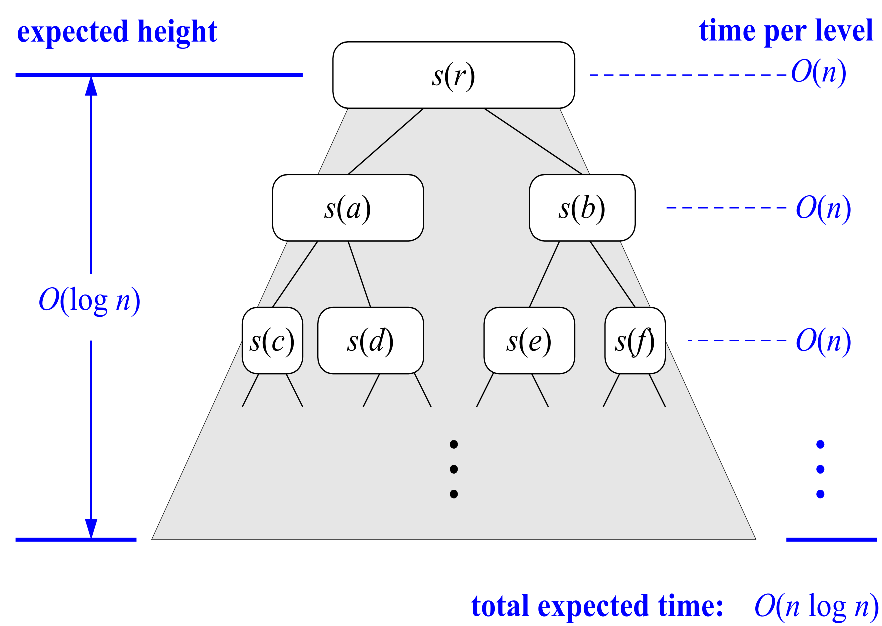
- $i/2$ ancestors are good calls
- size of the input sequence for the current call is at most $(3/4)^{i/2}\boldsymbol{n}$

For a node of depth $2\log_{4/3}\boldsymbol{n}$

- the expected input size is one
- the expected height of the quick-sort tree is $\boldsymbol{O}(\log \boldsymbol{n})$

The amount of work done at the nodes of the same depth is $\boldsymbol{O}(\boldsymbol{n})$

Thus, the expected running time of quick-sort is $\boldsymbol{O}(\boldsymbol{n}\log \boldsymbol{n})$



# In-Place Quick-Sort

During the partition step, use replace operations to rearrange elements of the input sequences such that:

- elements less than pivot have rank < ***h***
- elements equal to pivot have rank between [***h***, ***k***]
- elements greater than pivot have rank > ***k***

**Algorithm** ***inPlaceQuickSort***(***S, l, r***)

**Input** sequence ***S***, ranks ***l*** and ***r***

**Output** sequence ***S*** with the elements of rank between ***l*** and ***r*** rearranged in increasing order

**if** $l \geq r$

**return**

$i \leftarrow$ a random integer between ***l*** and ***r***

$x \leftarrow$ ***S.elemAtRank***(***i***)

(***h, k***) $\leftarrow$ ***inPlacePartition***(***x***)

***inPlaceQuickSort***(***S, l, h*** − 1)

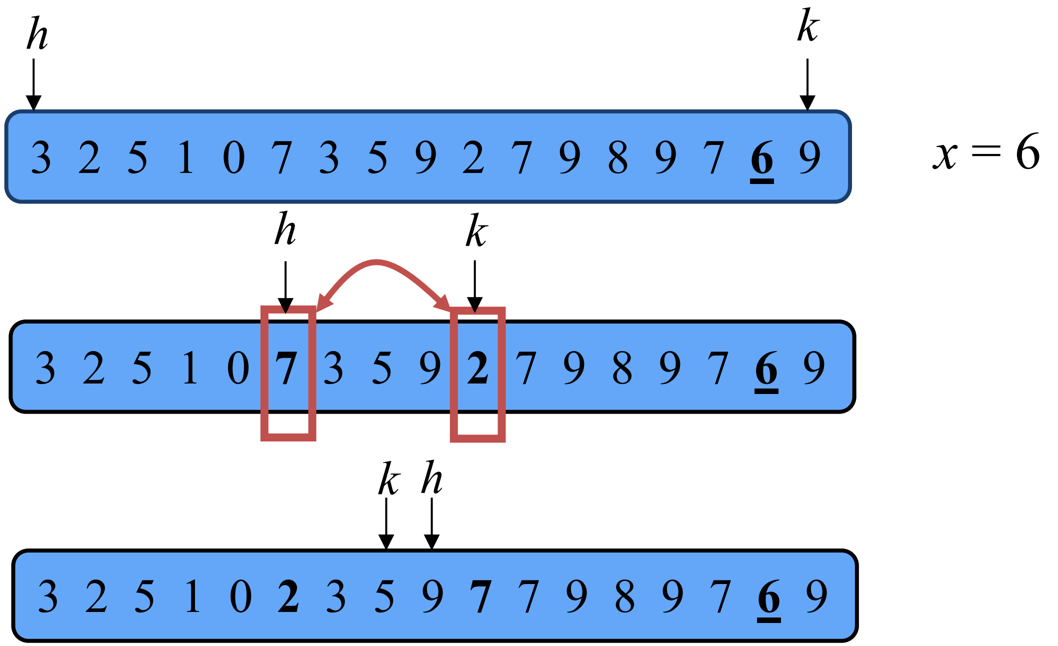
***inPlaceQuickSort***(***S, k*** + 1, ***r***)

# In-Place Partition

Performs a partitioning using two indices to split $S$ into $L$ and $E \cup G$ (a similar method can split $E \cup G$ into $E$ and $G$).

Repeat until $h$ and $k$ cross:

- Scan $h$ to the right until it finds an element $\geq x$
- Scan $k$ to the left until it finds an element $< x$
- Swap elements at indices $h$ and $k$



# Summary of Sorting Algorithms

| Algorithm | Time | Notes |
|---|---|---|
| selection-sort | $O(n^2)$ | ◆ in-place, not stable<br>◆ slow (good for small inputs) |
| insertion-sort | $O(n^2)$ | ◆ in-place, stable<br>◆ slow (good for small inputs) |
| quick-sort | $O(n \log n)$ expected | ◆ in-place, not stable<br>◆ randomized<br>◆ fast (good for large inputs) |
| heap-sort | $O(n \log n)$ | ◆ in-place, not stable<br>◆ fast (good for large inputs) |
| merge-sort | $O(n \log n)$ | ◆ not in-place, stable<br>◆ sequential data access<br>◆ fast (good for huge inputs) |

# Other: Nuts and Bolts

You are given a collection of $n$ bolts of different widths, and $n$ corresponding nuts.

- You can test whether a given nut and bolt fit together, from which you learn whether the nut is too large, too small, or an exact match for the bolt.
- The differences in size between pairs of nuts or bolts are too small to see by eye, so you cannot compare the sizes of two nuts or two bolts directly.
- You are to match each bolt to each nut.

Give an efficient algorithm to solve the nuts and bolts problem.