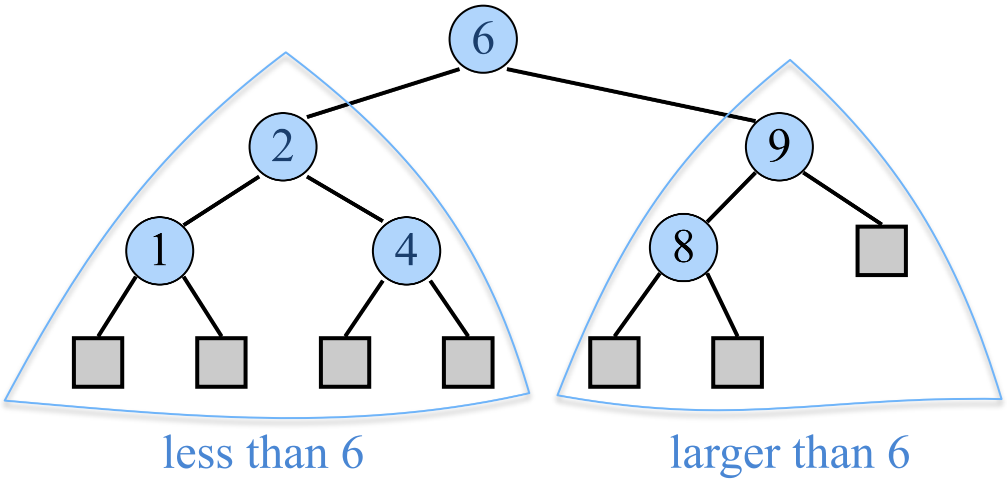
Binary Search Trees

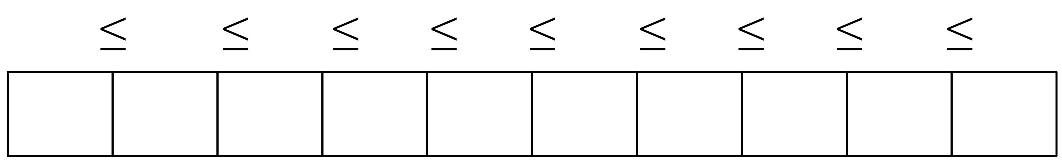


Ordered Dictionaries

- Keys are ordered
- Perform usual dictionary operations (insertItem, removeItem, findElement) and maintain an order relation for the keys
  - we use an external comparator for keys
- New operations:
  - closestKeyBefore(k), closestElemBefore(k)
  - closestKeyAfter(k), closestElemAfter(k)
- A special sentinel, NO_SUCH_KEY, is returned if no such item in the dictionary satisfies the query

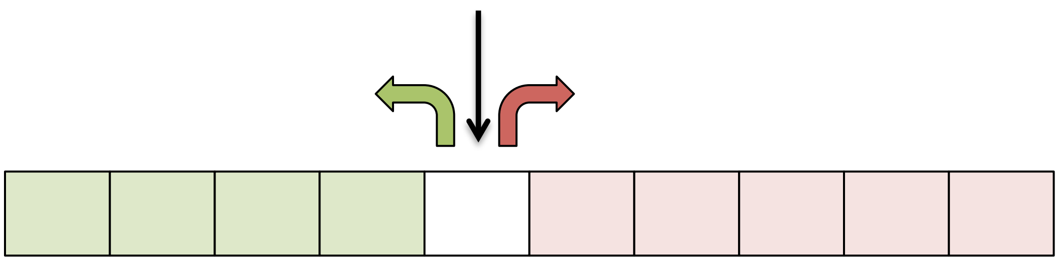
Binary Search

- Items are ordered in a sorted sequence
- Find an element *k*



Binary Search

- Items are ordered in a sorted sequence
- Find an element *k*
  - After checking a key *j* in the sequence, we can tell if item with key *k* will come before or after it



  - Which item should we compare against first?   The middle

Binary Search: Find k = 52

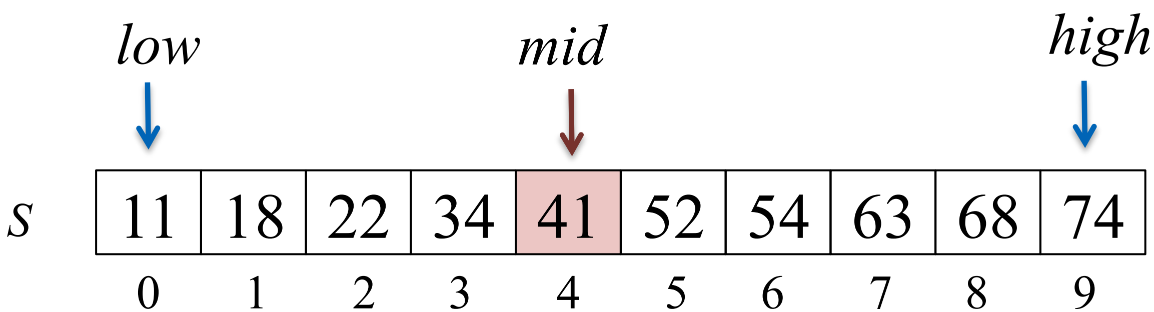
Algorithm BinarySearch(S, k, low, $high$):

	low									*high*
S	11	18	22	34	41	52	54	63	68	74
	0	1	2	3	4	5	6	7	8	9

Binary Search: Find k = 52

Algorithm BinarySearch(S, k, low, $high$):

$mid \leftarrow \lfloor (low + high) / 2 \rfloor$



Binary Search: Find k = 52

Algorithm BinarySearch(S, k, low, $high$):

$mid \leftarrow \lfloor (low + high) / 2 \rfloor$

if $key(mid) = k$ **then return** $elem(mid)$

	low				*mid*					*high*
S	11	18	22	34	41	52	54	63	68	74
	0	1	2	3	4	5	6	7	8	9

Binary Search: Find k = 52

Algorithm BinarySearch(S, k, low, $high$):

$mid \leftarrow \lfloor (low + high) / 2 \rfloor$

if $key(mid) = k$ **then return** $elem(mid)$

if $key(mid) > k$ **then return** $BinarySearch(S, k, mid + 1, high)$

						low ↓		*mid* ↓		*high* ↓
S	11	18	22	34	41	52	54	63	68	74
	0	1	2	3	4	5	6	7	8	9

Binary Search: Find k = 52

Algorithm BinarySearch(S, k, low, $high$):

$mid \leftarrow \lfloor (low + high) / 2 \rfloor$

if $key(mid) = k$ **then return** $elem(mid)$

if $key(mid) > k$ **then return** $BinarySearch(S, k, mid + 1, high)$

if $key(mid) < k$ **then return** $BinarySearch(S, k, low, mid - 1)$

mid

low *high*

S	11	18	22	34	41	52	54	63	68	74
	0	1	2	3	4	5	6	7	8	9

Binary Search: Find k = 52

Algorithm BinarySearch(S, k, low, $high$):

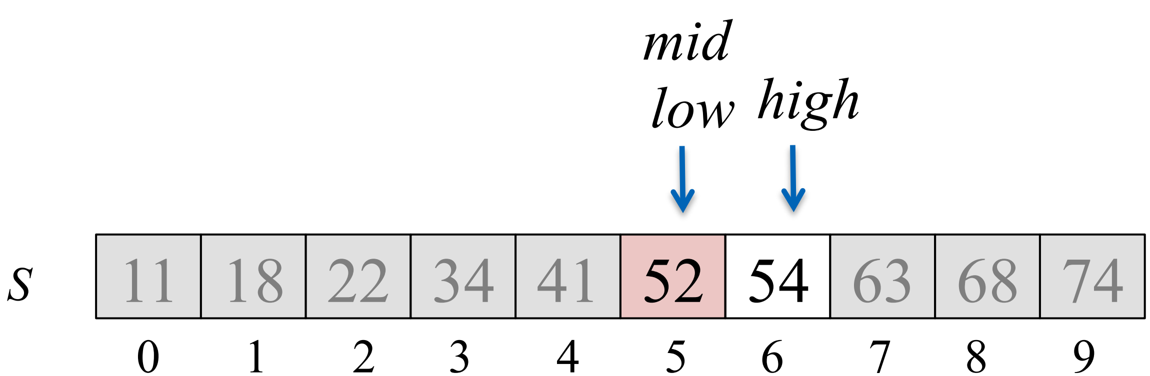
if $low > high$ **then return** *NO_SUCH_KEY*

$mid \leftarrow \lfloor (low + high) / 2 \rfloor$

if $key(mid) = k$ **then return** $elem(mid)$

if $key(mid) > k$ **then return** $BinarySearch(S, k, mid + 1, high)$

if $key(mid) < k$ **then return** $BinarySearch(S, k, low, mid - 1)$



Lookup Table

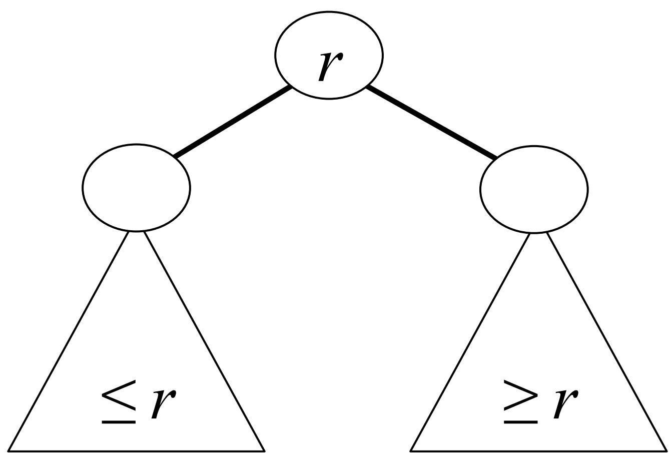
- A dictionary implemented by means of an array-based sequence which is sorted by key
  - why use an array-based sequence rather than a linked list?

- Performance:
  - insertItem takes $O(n)$ time to make room by shifting items
  - removeItem takes $O(n)$ time to compact by shifting items
  - findElement takes $O(\log n)$ time, using binary search

- Effective only for
  - small dictionaries, or
  - when searches are the most common operations, while insertions and removals are rarely performed

Binary Search Tree

- A binary search tree is a binary tree where each internal node stores a (key, element)-pair, and
  - each element in the left subtree is smaller than the root
  - each element in the right subtree is larger than the root
  - the left and right subtrees are binary search trees
- An inorder traversal visits items in ascending order



BST – Insert(k, v)

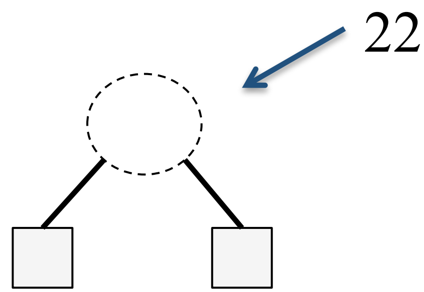
- Idea
  - find a free spot in the tree and add a node which stores that item (k, v)

- Strategy
  - start at root r
  - if k < key(r), continue in left subtree
  - if k > key(r), continue in right subtree
  - what if k = key(r) ?

- Runtime is $O(\boldsymbol{h})$, where $\boldsymbol{h}$ is the height of the tree

BST – Insert Example

Insert the numbers 22, 80, 18, 9, 90, 20.



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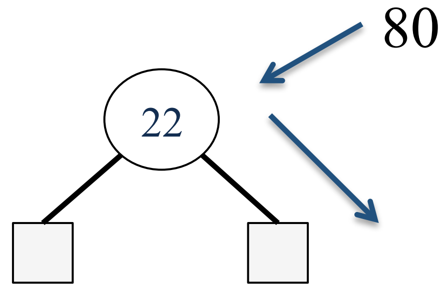

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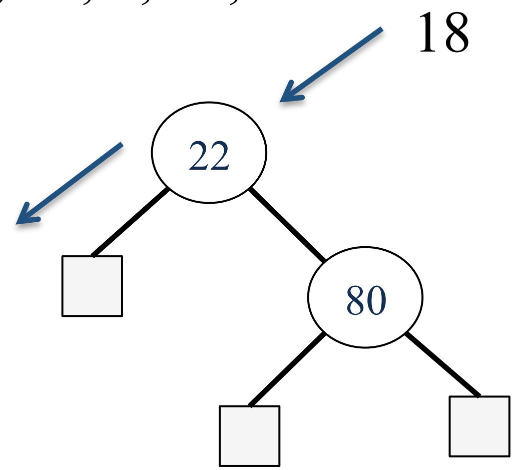

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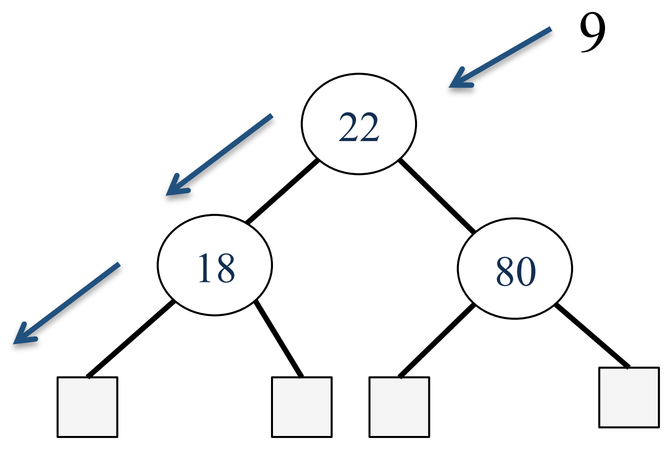

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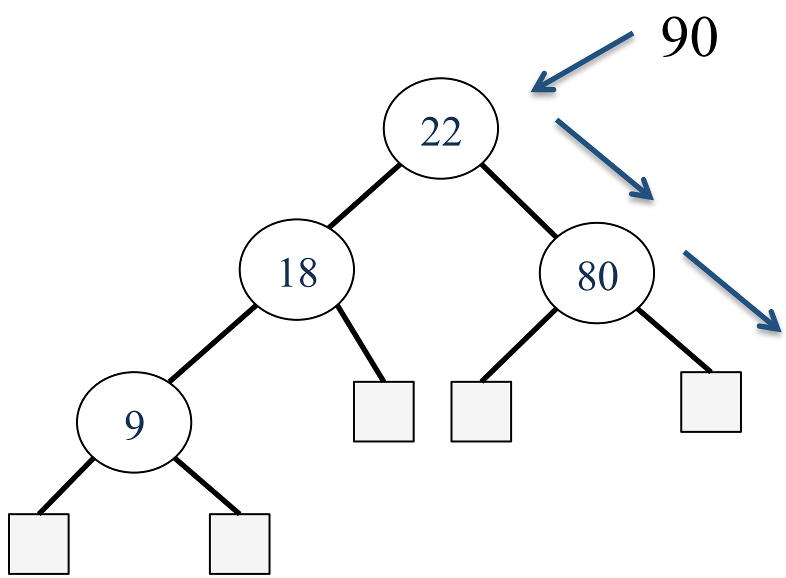

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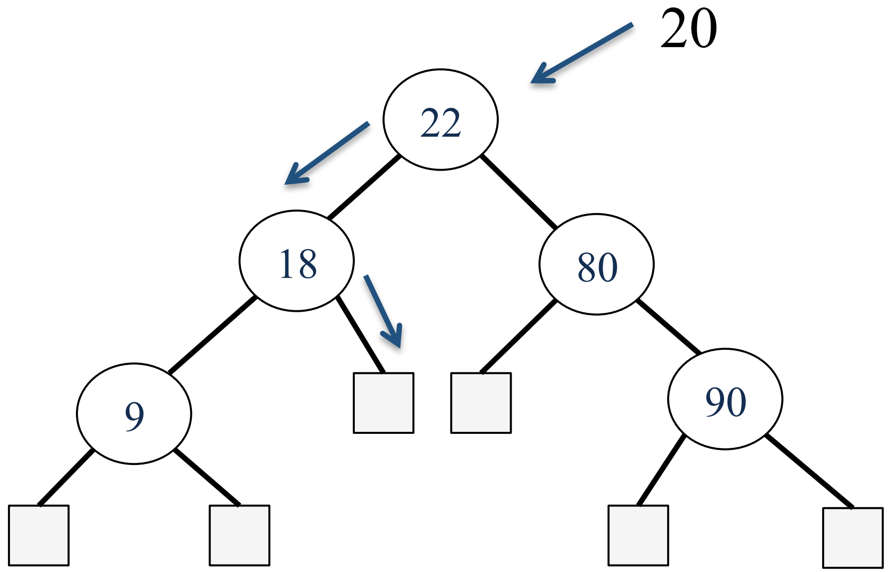

BST – Insert Example

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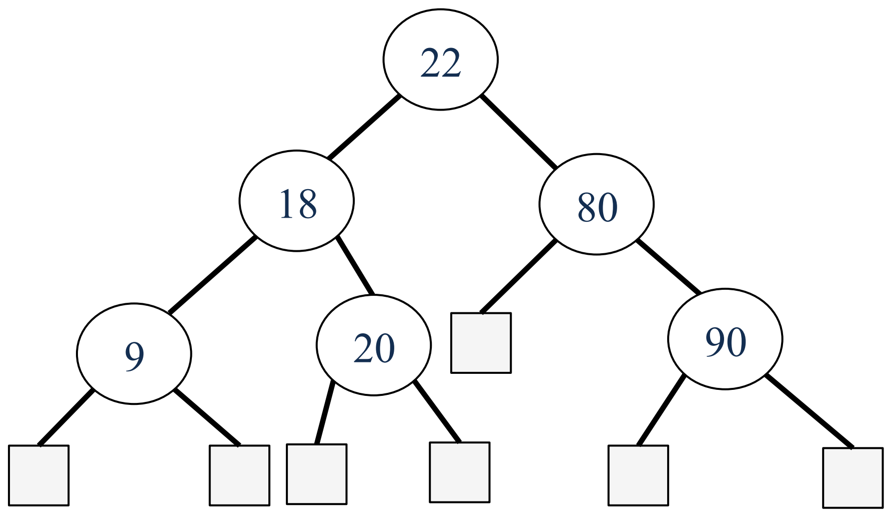


BST - Find

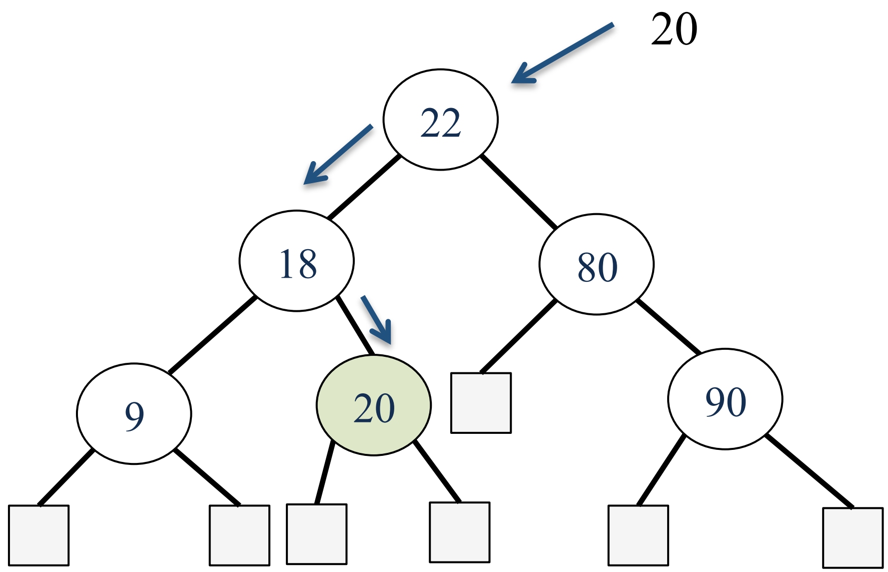
- Find the node with key k

- Strategy
  - start at root r
  - if k = key(r), return r
  - if k < key(r), continue in left subtree
  - if k > key(r), continue in right subtree

- Runtime is $O(h)$, where h is the height of the tree

BST – Find Example

Find the number 20



BST - Delete

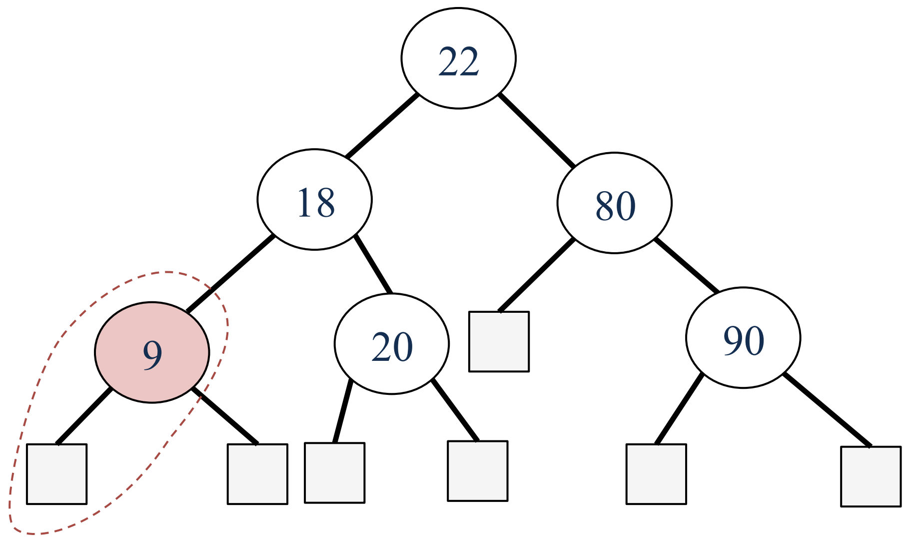
- Delete the node with key k

- Strategy: let n be the position of FindElement(k)
  - Remove n without creating "holes" in the tree
  - Case 0: n has two children with external nodes
  - Case 1: n has a child which is an internal node
  - Case 2: n has two children with internal nodes

- Runtime is $O(h)$, where h is the height of the tree

BST – Delete Example

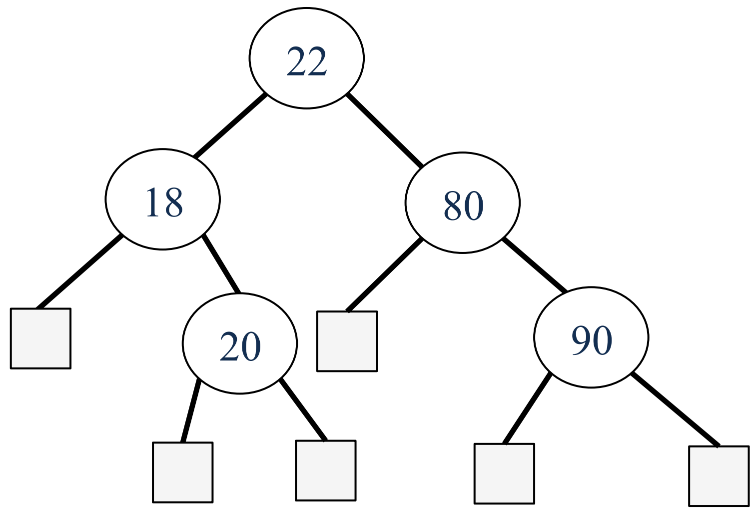
Case 0: *n* has two children which are external nodes



Delete 9

BST – Delete Example

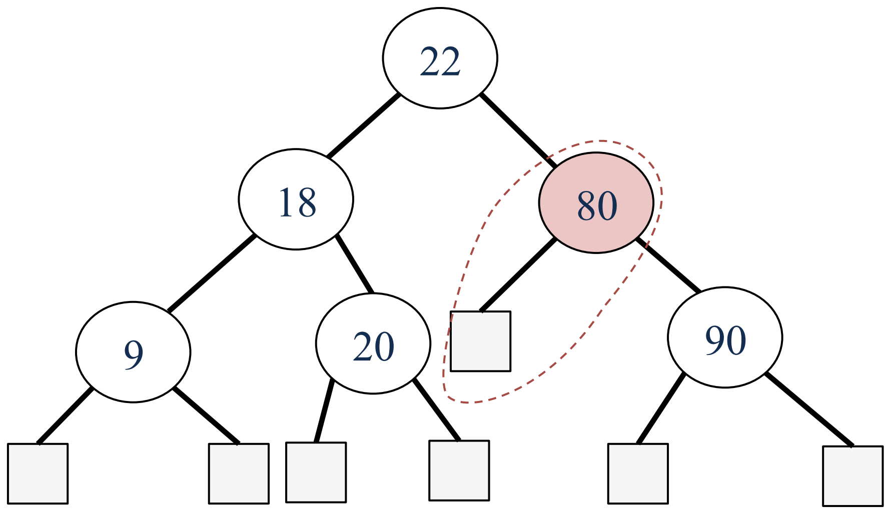
Case 0: *n* has two children which are external nodes



Delete 9

BST – Delete Example

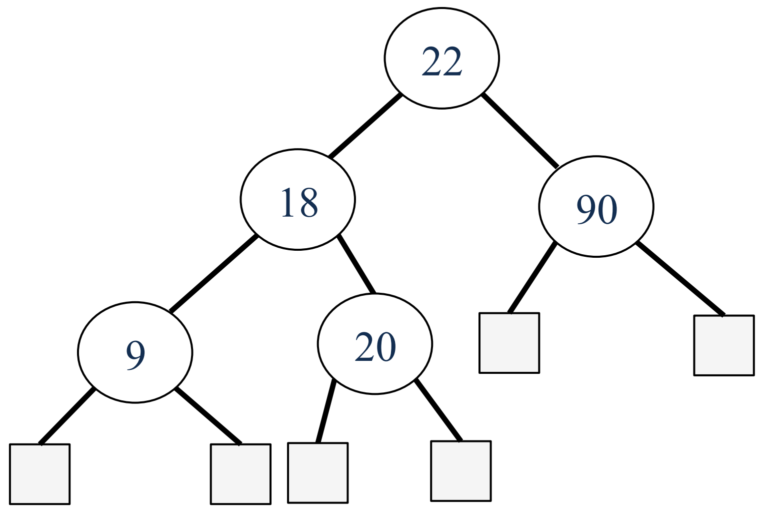
Case 1: *n* has a child which is an internal node



Delete 80

BST – Delete Example

Case 1: *n* has a child which is an internal node



Delete 80

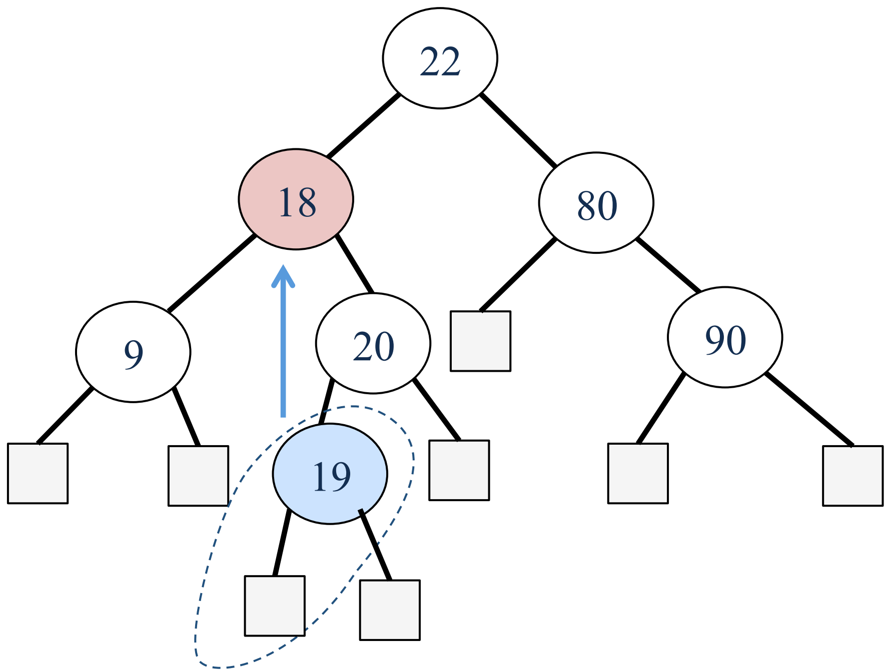
BST – Delete Example

Case 2: *n* has two children which are internal nodes

Find the first internal node *m* that follows *n* in an inorder traversal

We consider the subcase that *m* has two external nodes

Replace *n* with *m*



Delete 18

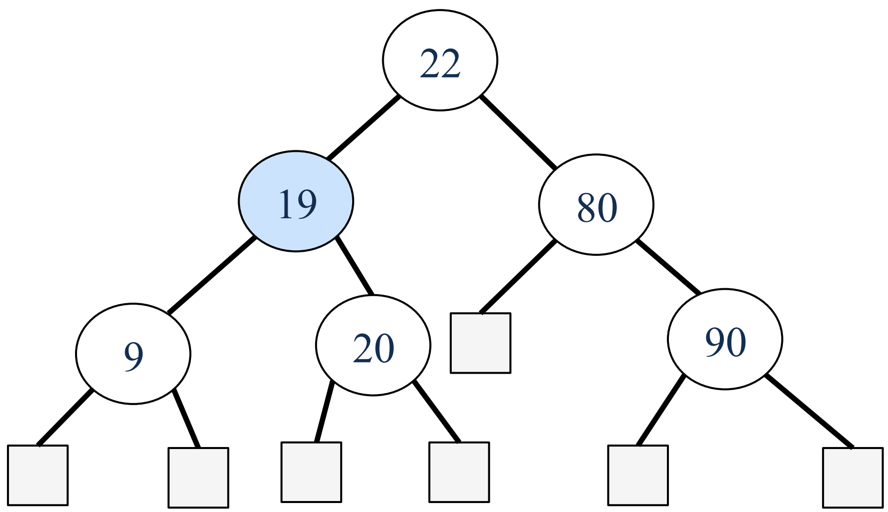
BST – Delete Example

Case 2: *n* has two children which are internal nodes

Find the first internal node *m* that follows *n* in an inorder traversal

We consider the subcase that *m* has two external nodes

Replace *n* with *m*



Delete 18

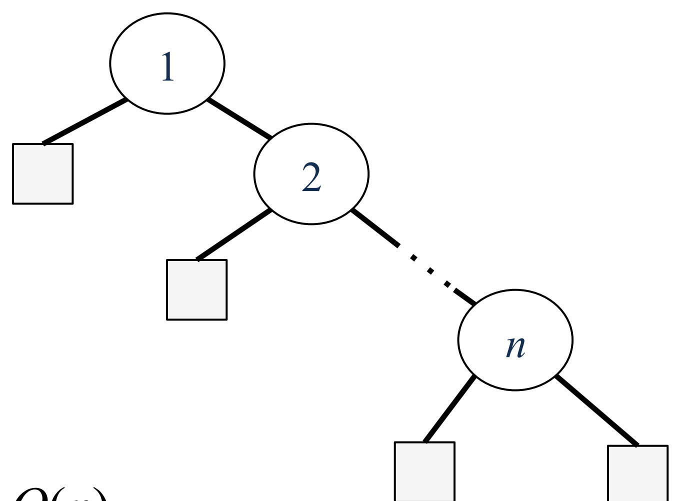
BST Performance

Space used is O(n)

Runtime of all operations is $O(h)$

- What is h in the worst case?

Consider inserting the sequence 1, 2, …, $n - 1$, n



Worst case height $h \in O(n)$.

- How do we keep the tree balanced?

Dictionary: Worst-case Comparison

	Unordered		**Ordered**		
	Log file	**Hash table**	**Lookup table**	**Binary Search Tree**	**Balanced Trees**
size, isEmpty	$O(1)$	$O(1)$	$O(1)$	$O(1)$	$O(1)$
keys, elements	$O(n)$	$O(n)$	$O(n)$	$O(n)$	$O(n)$
findElement	$O(n)$	$O(n)$**	$O(\log n)$	$O(h)$	$O(\log n)$
insertItem	$O(1)$	$O(n)$**	$O(n)$	$O(h)$	$O(\log n)$
removeElement	$O(n)$	$O(n)$**	$O(n)$	$O(h)$	$O(\log n)$
closestKey closestElem	$O(n)$	$O(n)$	$O(\log n)$	$O(h)$	$O(\log n)$

** Expected running time is $O(1)$

Other

- You are given two sorted integer arrays A and B such that no integer is contained twice in the same array. A and B are nearly identical. However, B is missing exactly one number. Find the missing number in B.

- You are given a sorted array A of distinct integers. Determine whether there exists an index i such that $A[i] = i$.

- You are given an array A of integers. Determine the most frequent number.