Priority Queues & Heaps

Priority Queue ADT

- Stores a collection of (key, element) pairs
- Main methods
  - insertItem(k, o): inserts an item with key k and element o
  - removeMin(): removes the item with smallest key and returns its element
  - minKey(): returns, but does not remove, the smallest key of an item
  - minElement(): returns, but does not remove, the element of an item with smallest key
  - size(), isEmpty()

- Applications:
  - Multithreading
  - Triage

Keys must be comparable

- Keys in a priority queue can be arbitrary objects on which an order is defined

- Two distinct items in a priority queue can have the same key

- Mathematical concept of total order relation $\leq$
  - Reflexive property: $x \leq x$
  - Antisymmetric property: $x \leq y \;\wedge\; y \leq x \;\Rightarrow\; x = y$
  - Transitive property: $x \leq y \;\wedge\; y \leq z \;\Rightarrow\; x \leq z$
  - Comparability: $x \leq y$ or $y \leq x$ for any x,y

Comparator ADT

- Encapsulates the action of comparing two objects according to a given total order relation
- A generic priority queue uses an auxiliary comparator
- The comparator is external to the keys being compared
- When the priority queue needs to compare two keys, it uses its comparator
- Predicate methods:
  - isLessThan(x, y)
  - isLessThanOrEqualTo(x,y)
  - isEqualTo(x,y)
  - isGreaterThan(x, y)
  - isGreaterThanOrEqualTo(x,y)
  - isComparable(x)

Suppose you are given a priority queue implementation, so you have the following operations to work with:

insertItem(k, o)
removeMin()
minKey()
minElement()
size()
isEmpty()

How can you use it to sort a sequence ***S*** of numbers?

Sorting with a Priority Queue

We can use a priority queue to sort a set of comparable elements

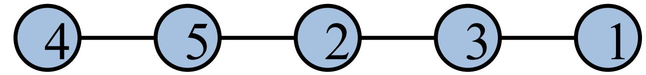
1. Insert the elements one by one with a series of insertItem(e, e) operations
2. Remove the elements in sorted order with a series of removeMin() operations

Running time depends on the priority queue implementation

Algorithm *PQ-Sort*(*S*, *C*)
    Input sequence ***S***, comparator ***C*** for the elements of ***S***
    Output sequence ***S*** sorted in increasing order according to ***C***
    $P \leftarrow$ priority queue with comparator ***C***
    while ¬*S.isEmpty* ()
        $e \leftarrow$ *S.remove* (*S. first* ())
        P.insertItem(*e*, *e*)
    while ¬*P.isEmpty*()
        $e \leftarrow$ *P.removeMin*()
        S.insertLast(*e*)

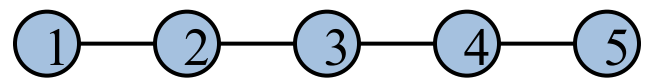
Sequence-based Priority Queue

Implementation with an unsorted sequence



- Store the items of the priority queue in a list-based sequence, in arbitrary order
- insertItem takes $O(1)$ time since we can insert the item at the beginning or end of the sequence
- removeMin, minKey and minElement take $O(n)$ time since we have to traverse the entire sequence to find the smallest key

Implementation with a sorted sequence



- Store the items of the priority queue in a sequence, sorted by key
- insertItem takes $O(n)$ time since we have to find the place where to insert the item
- removeMin, minKey and minElement take $O(1)$ time since the smallest key is at the beginning of the sequence

Selection-Sort

- Selection-sort is the variation of PQ-sort where the priority queue is implemented with an unsorted sequence

- Running time of Selection-sort:
  1. Inserting the elements into the priority queue with n insertItem operations takes $O(n)$ time
  2. Removing the elements in sorted order from the priority queue with n removeMin operations takes time proportional to

$$1 + 2 + \ldots + n$$

- Runs in $O(n^2)$ time

Insertion-Sort

- Insertion-sort is the variation of PQ-sort where the priority queue is implemented with a sorted sequence

- Running time of Insertion-sort:
  1. Inserting the elements into the priority queue with $\boldsymbol{n}$ insertItem operations takes time proportional to

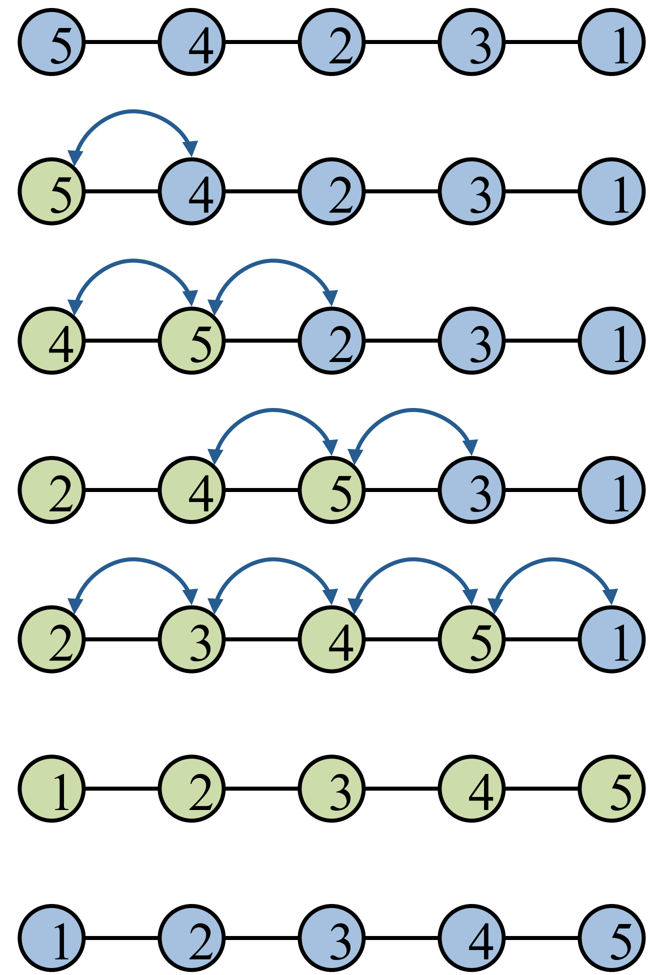
     $$1 + 2 + \ldots + \boldsymbol{n}$$

  2. Removing the elements in sorted order from the priority queue with a series of $\boldsymbol{n}$ removeMin operations takes $\boldsymbol{O}(\boldsymbol{n})$ time

- Runs in $\boldsymbol{O}(\boldsymbol{n}^2)$ time

In-place Insertion-sort

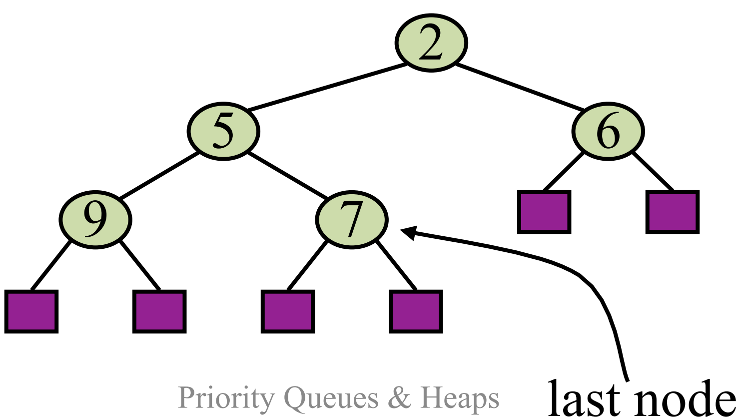
- Instead of using an external data structure, we can implement selection-sort and insertion-sort in-place
- A portion of the input sequence itself serves as the priority queue
- For in-place insertion-sort
  - We keep sorted the initial portion of the sequence
  - We can use swapElements instead of modifying the sequence



What is a Heap

- A heap is a binary tree storing keys at its internal nodes and satisfying the following properties:
  - Heap-Order: for every internal node v other than the root,
    $key(v) \geq key(parent(v))$
  - Complete Binary Tree: let h be the height of the heap
    - for $i = 0, \ldots, h - 1$, there are 2^i nodes of depth i
    - at depth $h - 1$, the internal nodes are to the left of the external nodes

- The last node of a heap is the rightmost internal node of depth $h - 1$

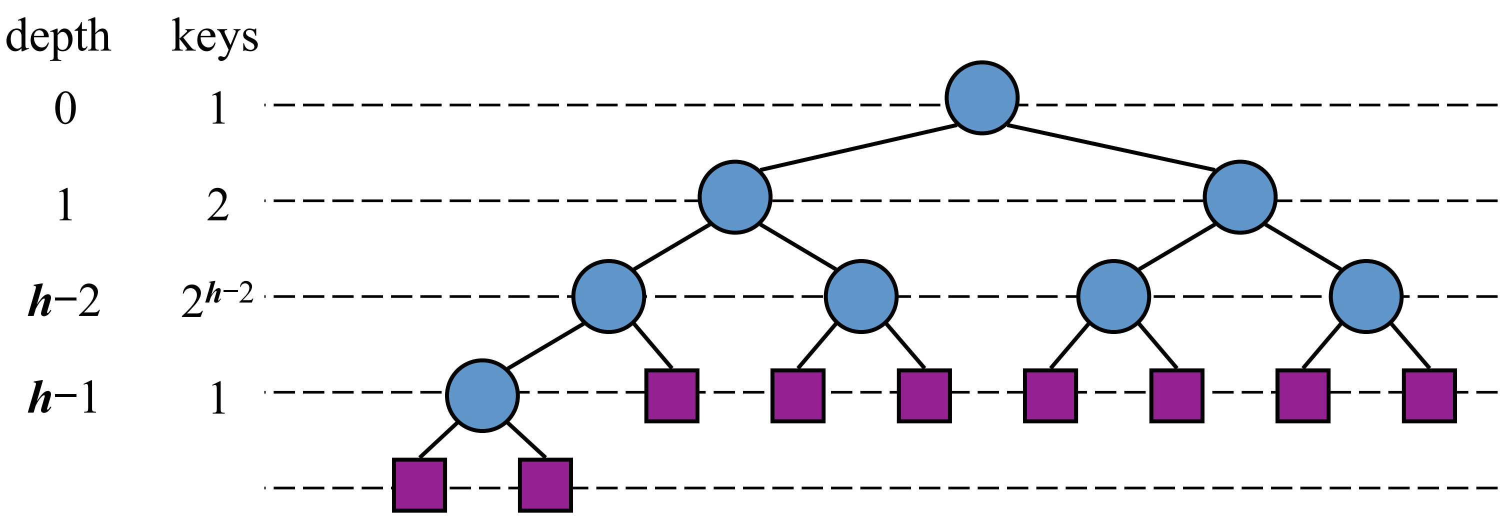


Height of a Heap

Theorem: A heap storing $\boldsymbol{n}$ keys has height $\boldsymbol{O}(\log \boldsymbol{n})$

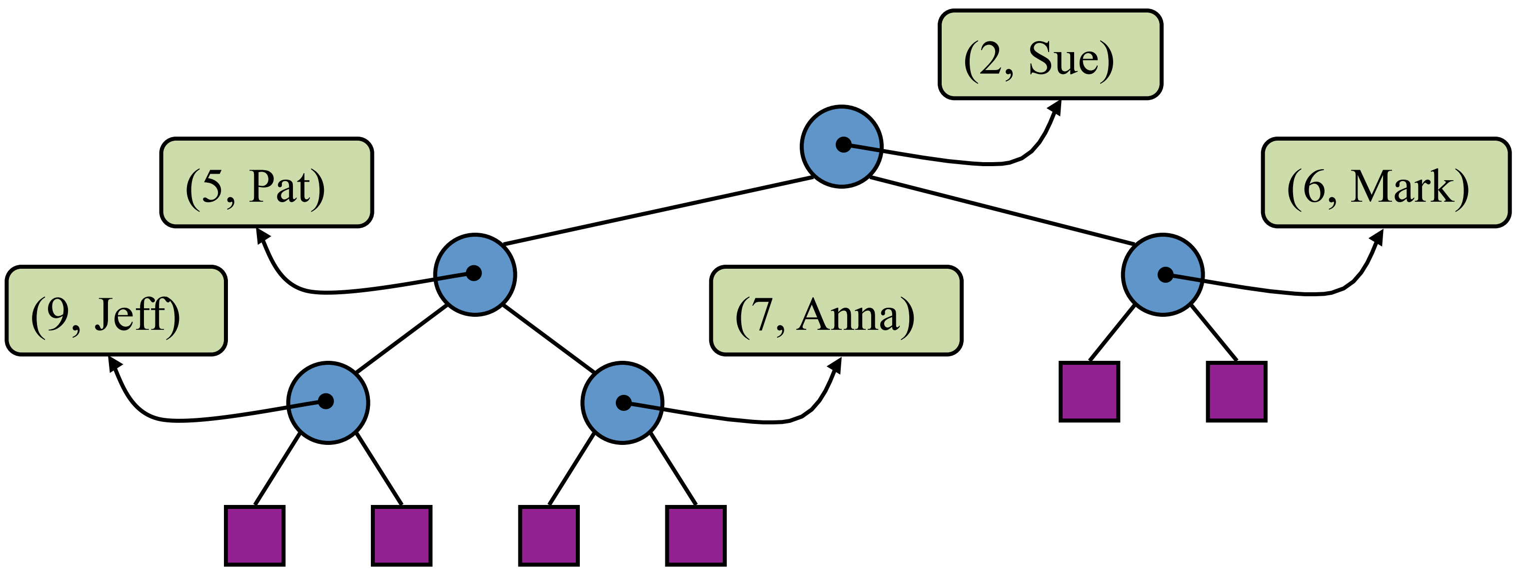
Proof: (we apply the complete binary tree property)

- Let $\boldsymbol{h}$ be the height of a heap storing $\boldsymbol{n}$ keys
- Since there are 2^i keys at depth $\boldsymbol{i} = 0, \ldots, \boldsymbol{h} - 2$ and at least one key at depth $\boldsymbol{h} - 1$, we have $\boldsymbol{n} \geq 1 + 2 + 4 + \ldots + 2^{\boldsymbol{h}-2} + 1$
- Thus, $\boldsymbol{n} \geq 2^{\boldsymbol{h}-1}$, i.e., $\boldsymbol{h} \leq \log \boldsymbol{n} + 1$



Heaps and Priority Queues

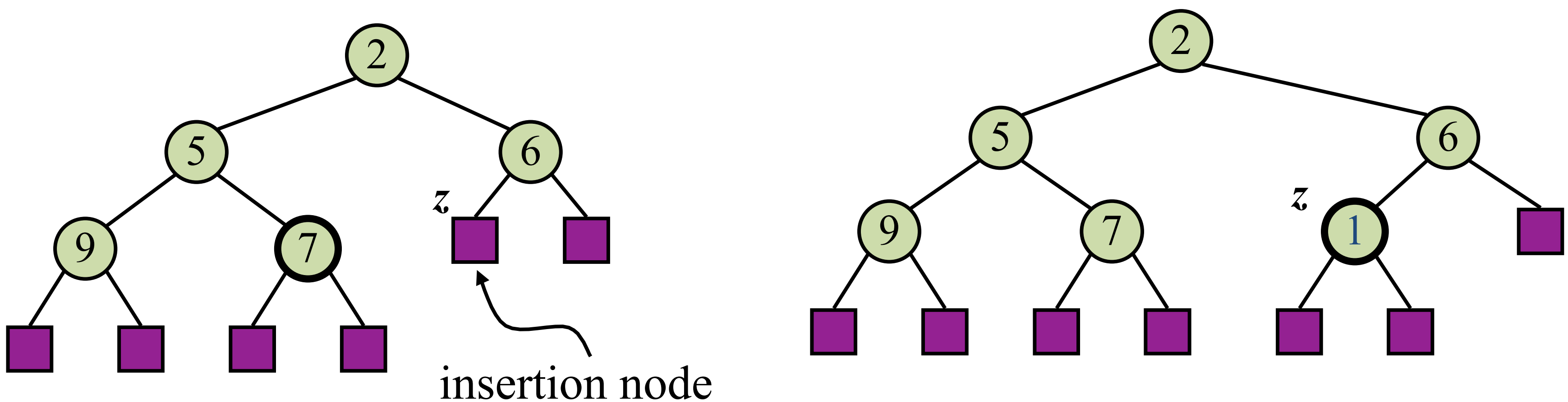
- We can use a heap to implement a priority queue
- We store a (key, element) item at each internal node
- We keep track of the position of the last node
- For simplicity, we show only the keys in the pictures



Insertion into a Heap: insertItem(*k*,*o*)

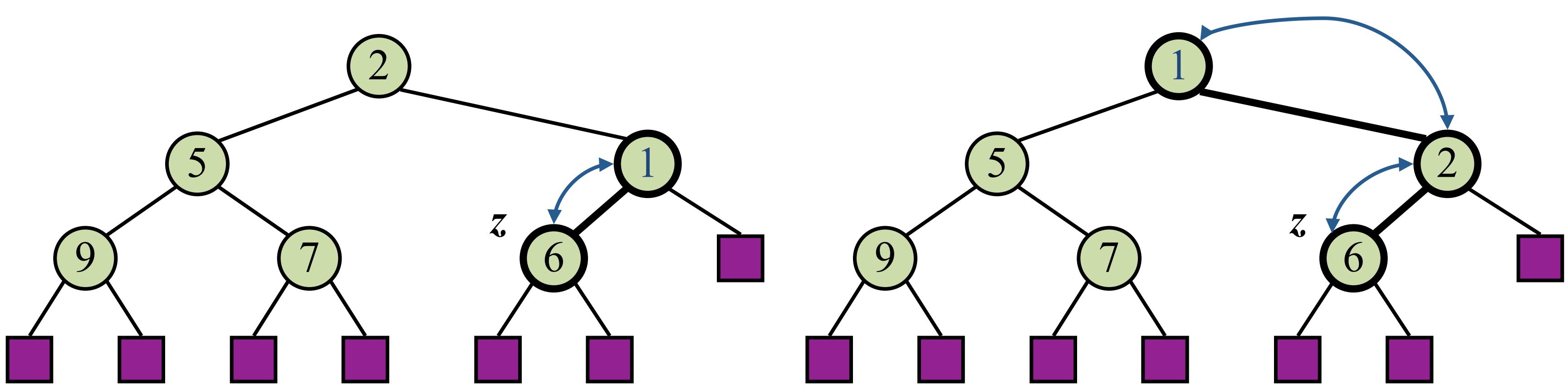
Consists of three steps:

- Find the insertion node ***z*** (the new last node)
- Store ***k*** at ***z*** and expand ***z*** into an internal node
- Restore the heap-order property (discussed next)



Upheap Bubbling

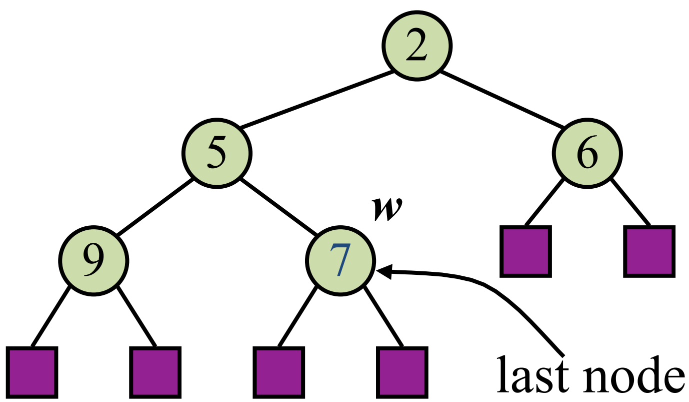
- After the insertion of a new key ***k***, the heap-order property may be violated
- Algorithm upheap restores the heap-order property by swapping ***k*** along an upward path from the insertion node
- Terminates when the key ***k*** reaches the root or a node whose parent has a key smaller than or equal to ***k***
- Since a heap has height ***O***(log ***n***), upheap runs in ***O***(log ***n***) time

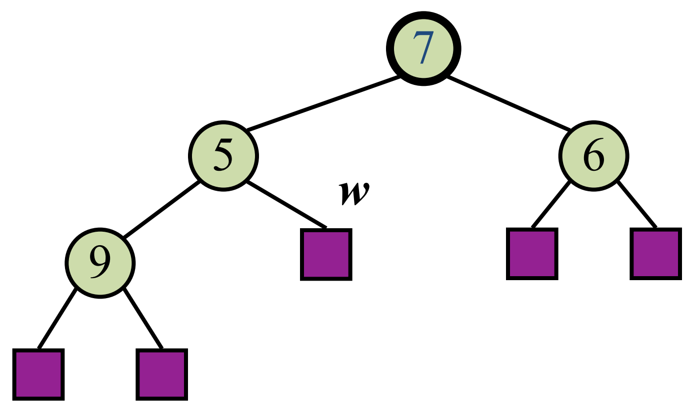


Removal from a Heap: removeMin()

Consists of three steps

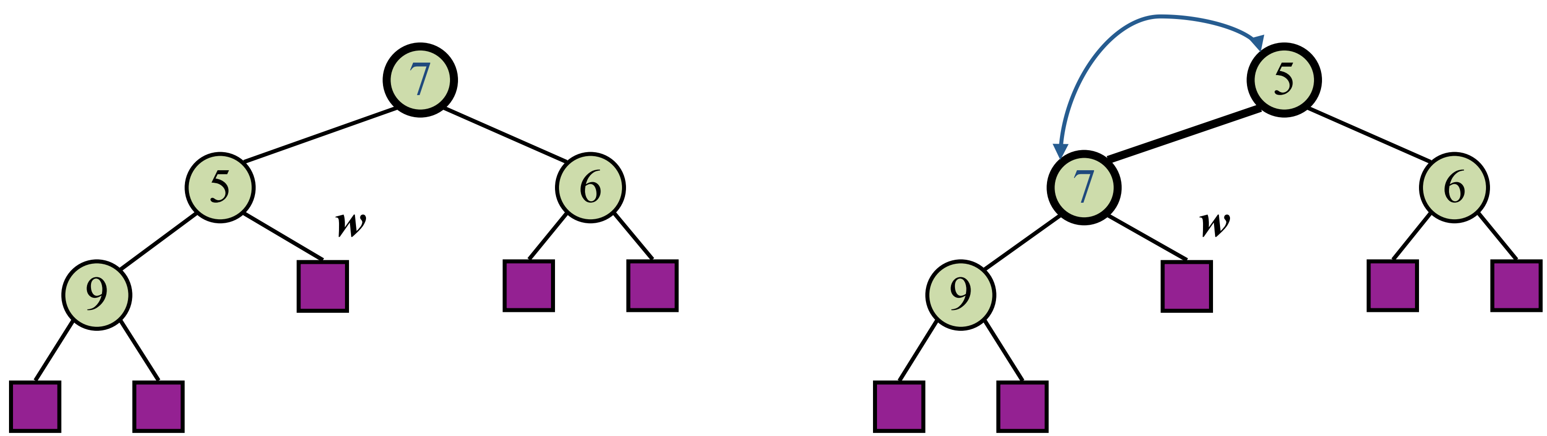
- Replace the root key with the key of the last node ***w***
- Compress ***w*** and its children into a leaf
- Restore the heap-order property (discussed next)





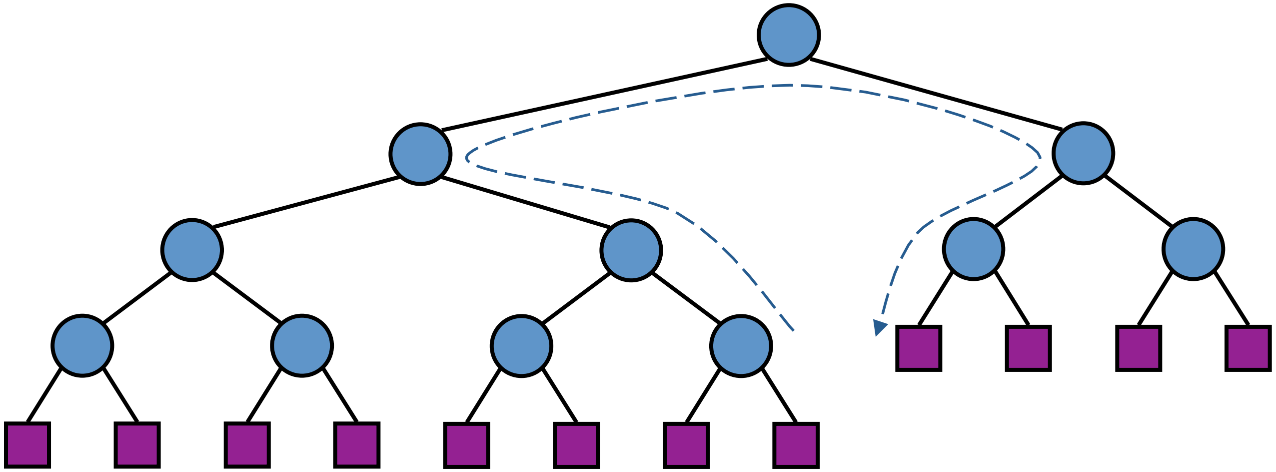
Downheap Bubbling

- After replacing the root key with the key ***k*** of the last node, the heap-order property may be violated
- Algorithm downheap restores the heap-order property by swapping key ***k*** with the smallest key among children along a downward path from the root
- Terminates when key ***k*** reaches a leaf or a node whose children have keys greater than or equal to ***k***
- Since a heap has height ***O***(log ***n***), downheap runs in ***O***(log ***n***) time



Finding the Last Node

- The last node can be found by traversing a path of $O(\log n)$ nodes
  - While the current node is a right child, go to the parent node
  - If the current node is a left child of v, go to the right child of v
  - While the current node is internal, go to the left child
- Similar algorithm for updating the last node after a removal



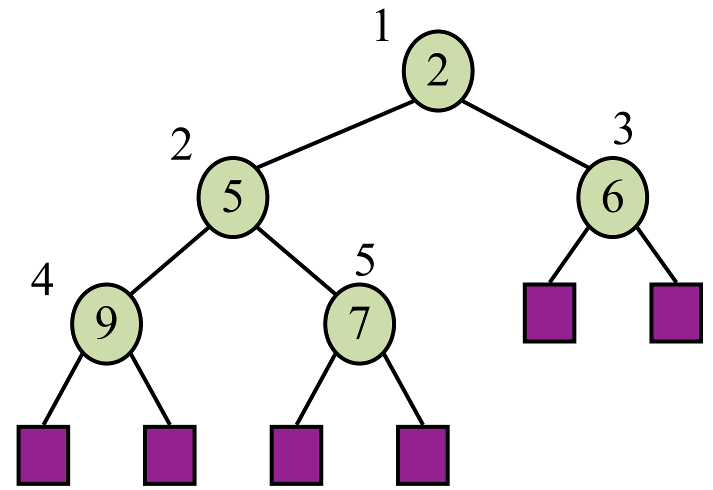
Heap-Sort

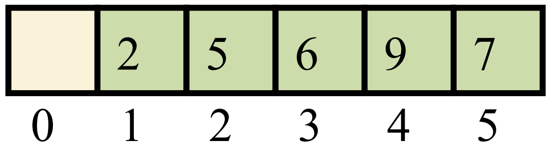
- Consider a priority queue with n items implemented by means of a heap
  - the space used is $O(n)$
  - methods insertItem and removeMin take $O(\log n)$ time
  - methods size, isEmpty, minKey, and minElement take time $O(1)$ time
- Using a heap-based priority queue, we can sort a sequence of n elements in $O(n \log n)$ time
  - much faster than quadratic sorting algorithms, such as insertion-sort and selection-sort

Vector-based Heap Implementation

We can represent a heap with $\boldsymbol{n}$ keys by means of a vector of length $\boldsymbol{n} + 1$

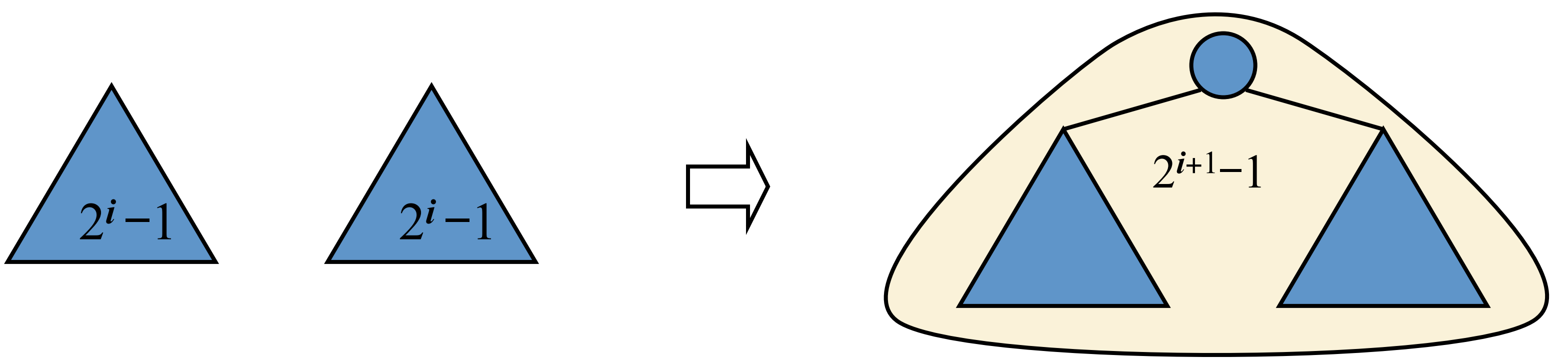
- For the node at rank $\boldsymbol{i}$
  - left child is at rank $2\boldsymbol{i}$
  - right child is at rank $2\boldsymbol{i} + 1$
- What does not need to be stored:
  - links between nodes
  - leaves
- The cell at rank 0 is not used
- Last node is at rank $\boldsymbol{n}$
  - insertItem inserts at rank $\boldsymbol{n} + 1$
  - removeMin removes at rank $\boldsymbol{n}$ (after swapping root with last node)
- Yields in-place heap-sort





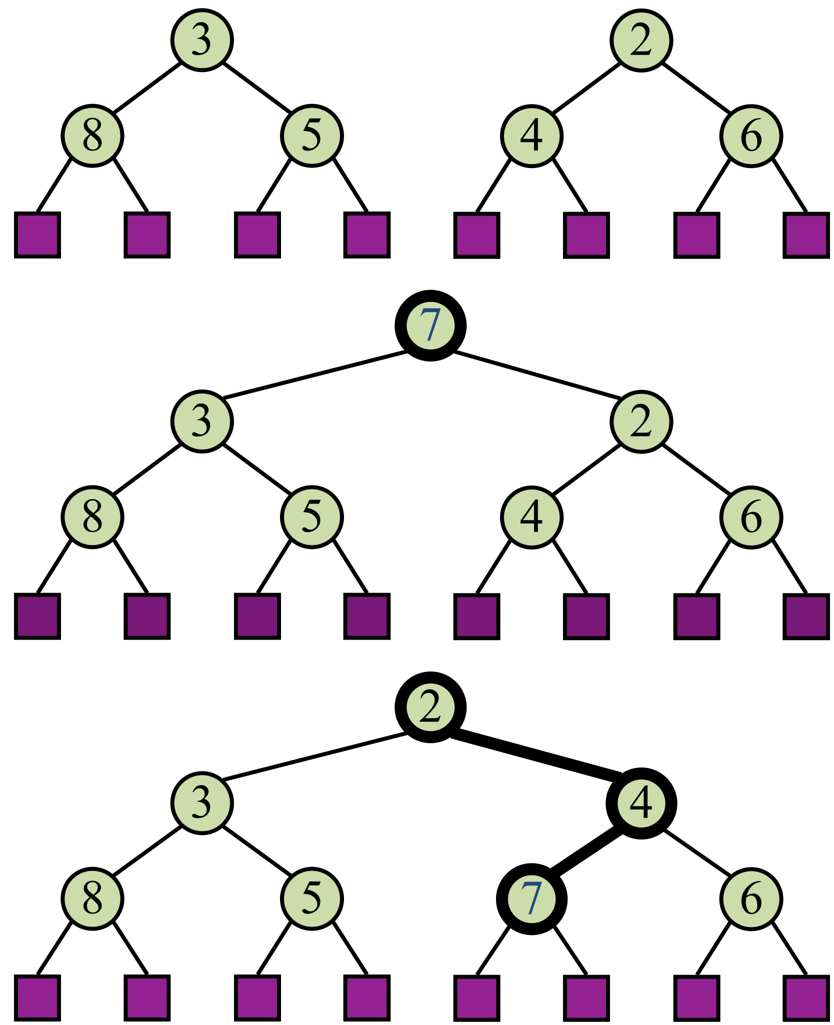
Bottom-up Heap Construction

- If all keys are known in advance, we can build a heap recursively
- For simplicity, assume number of keys $\boldsymbol{n} = 2^h - 1$ so the heap is a complete binary tree with every level being full
- Given $\boldsymbol{n}$ keys, build heap using a bottom-up construction with log $\boldsymbol{n}$ phases
- In phase $\boldsymbol{i}$, pairs of heaps with $2^{\boldsymbol{i}} - 1$ keys are merged into heaps with $2^{\boldsymbol{i}+1} - 1$ keys

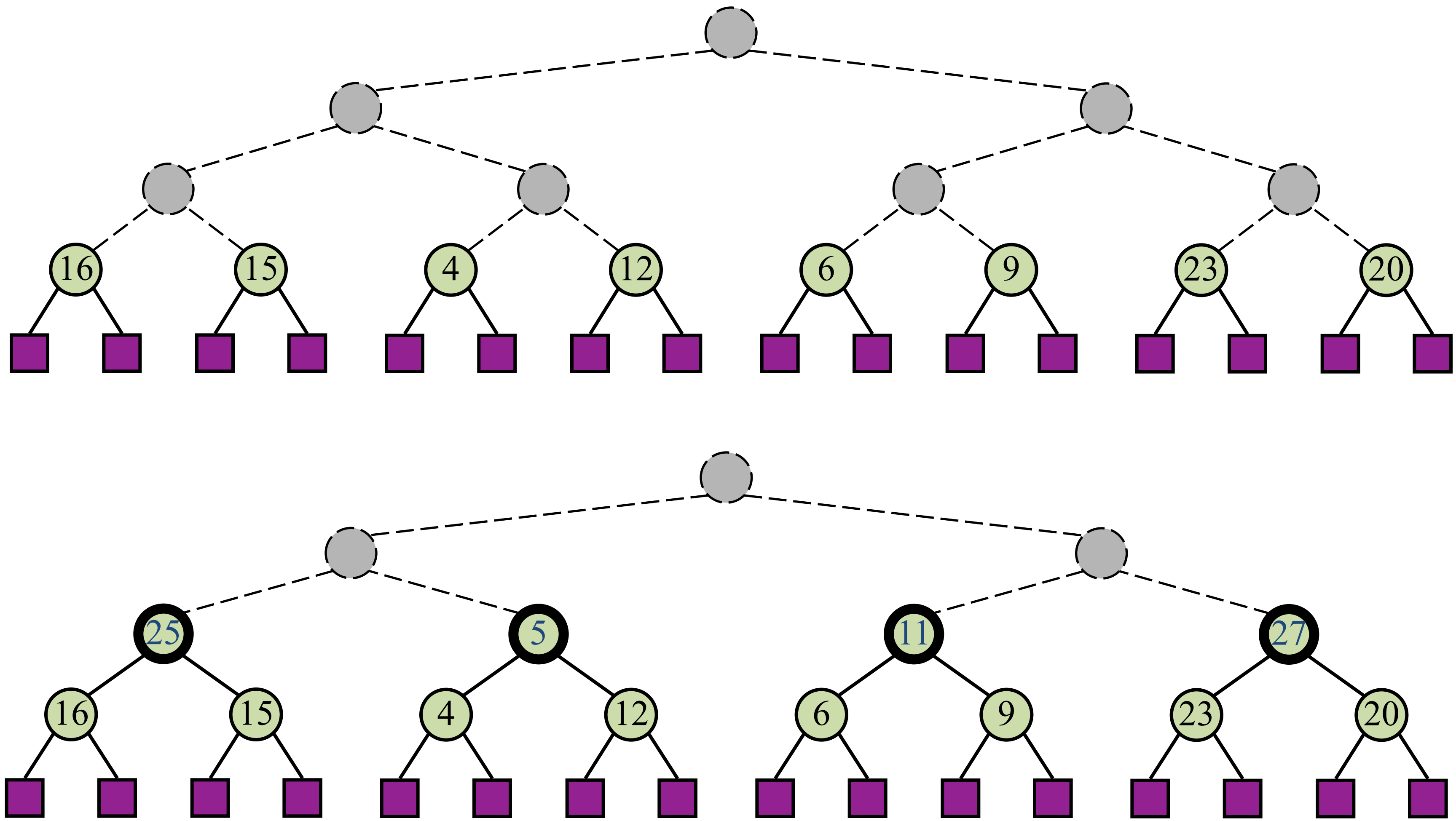


Merging Two Heaps

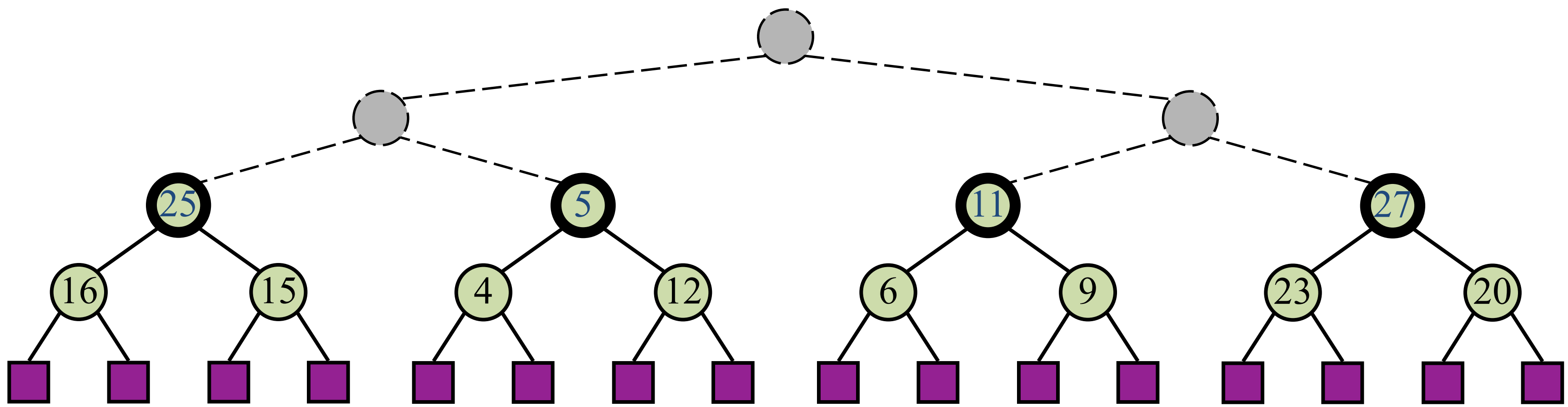
- We are given two heaps and a key ***k***
- We create a new heap with the root node storing ***k*** and with the two heaps as subtrees
- We perform downheap to restore the heap-order property

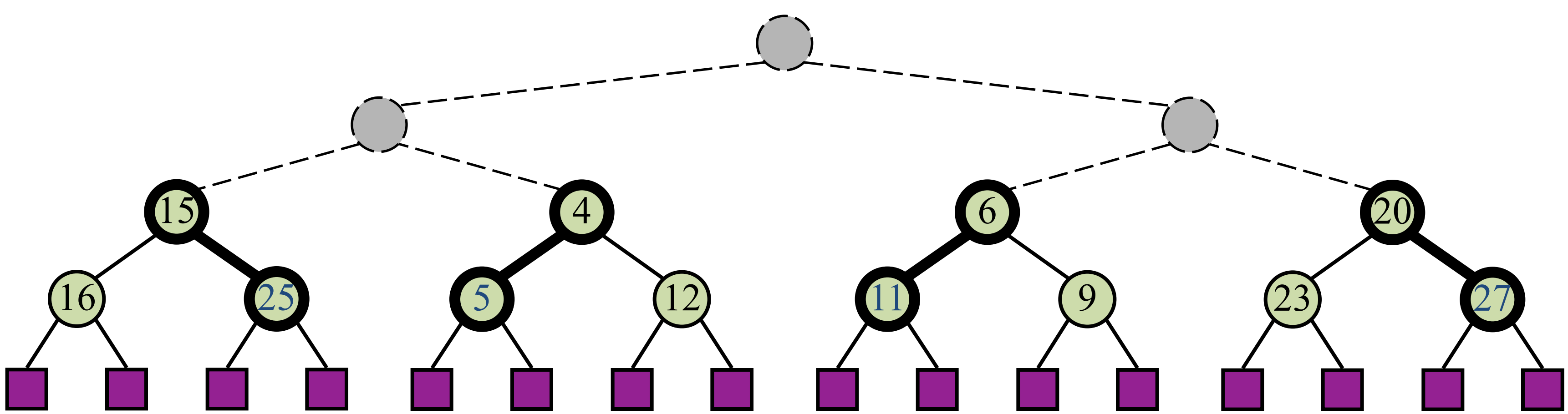


Example



Example (contd.)





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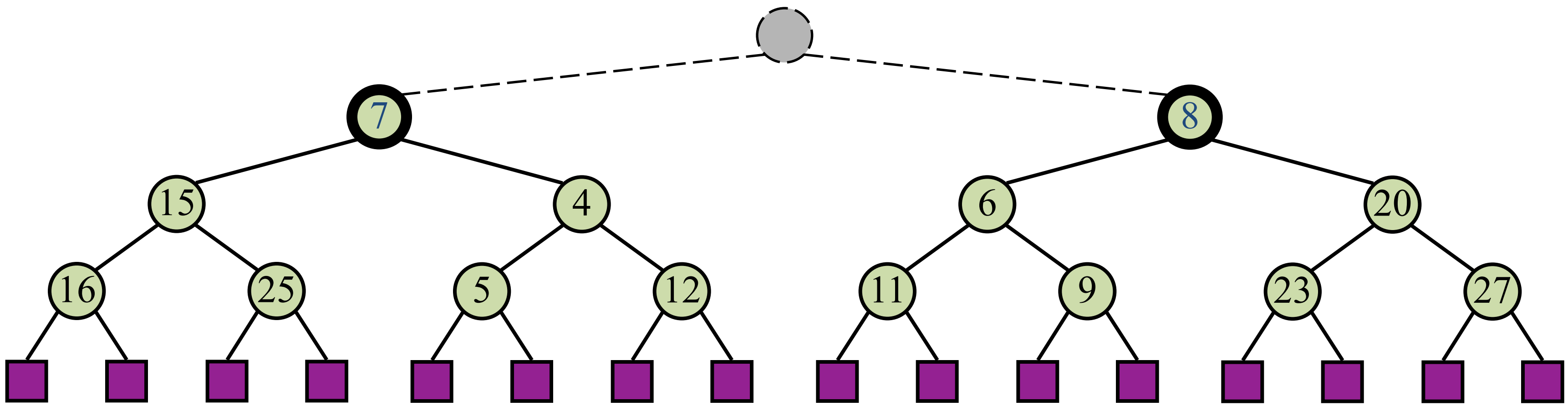
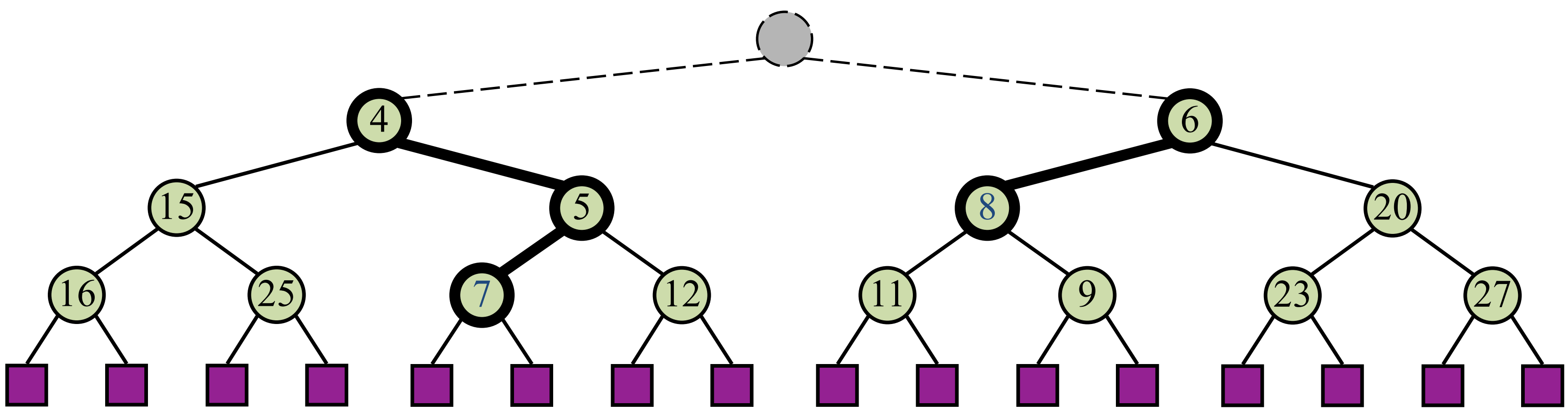
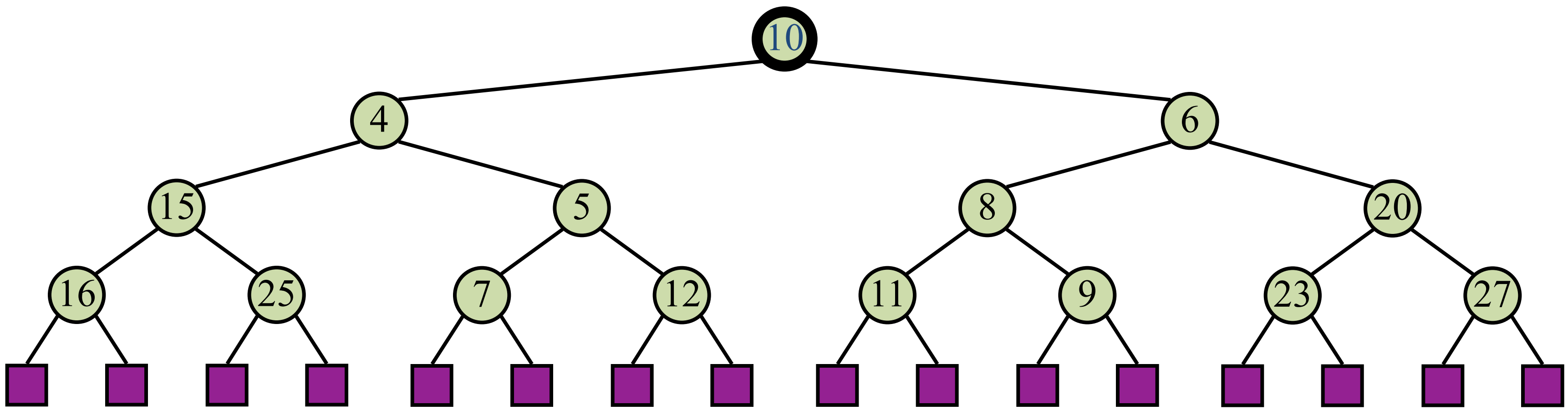
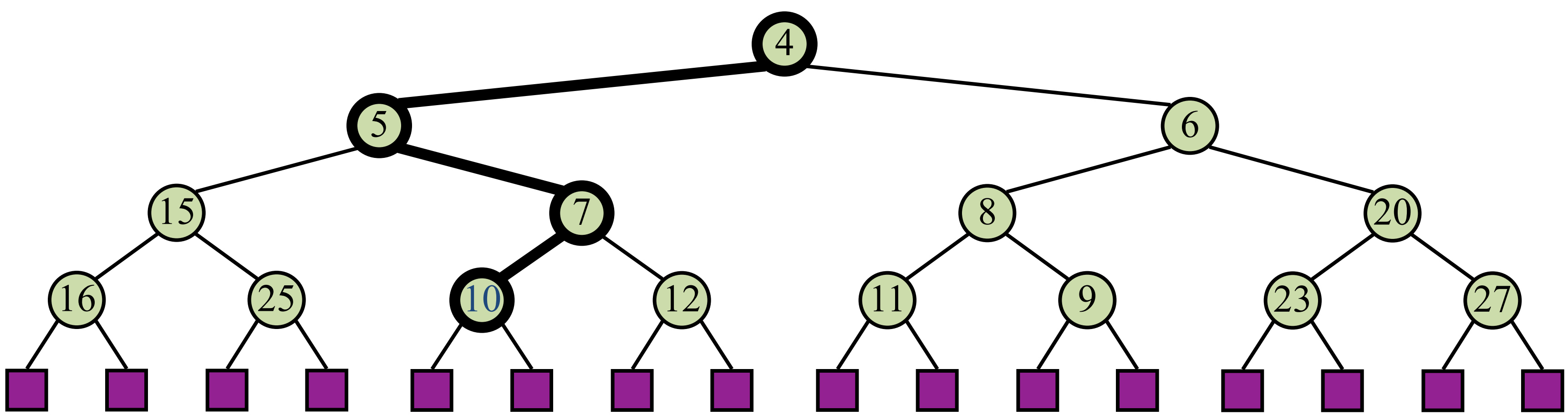




Example (end)





Analysis

- We visualize the worst-case time of a downheap with a proxy path that goes first right and then repeatedly goes left until the bottom of the heap (this path may differ from the actual downheap path)
- Since each node is traversed by at most two proxy paths, the total number of nodes of the proxy paths is $O(n)$
- Thus, bottom-up heap construction runs in $O(n)$ time
- Bottom-up heap construction is faster than n successive insertions and speeds up the first phase of heap-sort