Fast Deterministic Algorithms for Computing All Eccentricities in (Hyperbolic) Helly Graphs

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Outline

1. Preliminaries
   - Vertex eccentricities
   - Hyperbolic graphs
   - Helly graphs

2. Our contribution
   - Characterization of Helly graphs
   - Characterization of hyperbolicity in Helly graphs
   - All eccentricities in Helly graphs

Importance of vertex eccentricities

The *eccentricity* $e(v)$ of a vertex v is the maximum distance from v to any other vertex.

- *Radius* $rad(G)$ is the smallest eccentricity of vertices
- *Diameter* $diam(G)$ is the largest eccentricity of vertices
- A *central vertex* v is a vertex with minimum eccentricity, i.e., $e(v) = rad(G)$

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Applications:

- Vertex eccentricity is used to measure the importance of a node in the network: the eccentricity centrality index of v is defined as $1/e(v)$
- Diameter and radius play an important role in design and analysis of networks
- Finding a central vertex is a famous facility location problem

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Computing diameter and/or radius on a graph with n vertices and m edges – **in general**:

- Naïve approach in $O(nm)$: run breadth-first search from each vertex to compute its eccentricity, then take minimum (radius) and maximum (diameter)
- Conditionally optimal for general graphs and some restricted families
- Under plausible complexity assumptions, neither diameter nor radius can be computed in truly subquadratic time

Computing diameter and/or radius on a graph with n vertices and m edges – **in particular**

Tailored approach: exploit geometric and tree-like representations and/or some forbidden pattern (e.g., excluding a minor, or a family of induced subgraphs).

Examples:

- $O(m)$ algorithm to compute radius and central vertex of chordal graph
- Efficient algorithms for computing diameter and/or radius or finding a central vertex exist for
  - interval graphs
  - AT-free graphs
  - directed path graphs
  - distance-hereditary graphs
  - strongly chordal graphs
  - dually chordal graphs
  - chordal bipartite graphs
  - outerplanar graphs
  - planar graphs
  - Graphs with bounded clique-width or tree-width
  - H-minor free graphs
  - Graphs of bounded (distance) VC-dimension

δ-Hyperbolic graphs

Definition (Hyperbolicity)

The *hyperbolicity* is the smallest half-integer $\delta \geq 0$ such that, for any four vertices u, v, w, x, the two largest of the three distance sums $d(u, v) + d(w, x)$, $d(u, w) + d(v, x)$, $d(u, x) + d(v, w)$ differ by at most 2δ.

- As tree-width measures a graph's combinatorial tree-likeness, so does the hyperbolicity measure its metric tree-likeness
- Examples:
  - trees and complete graphs have $\delta = 0$
  - a cycle C_{4k} has $\delta = k$.
- Complexity to compute:
  - Exactly in $O(n^{3.69})$ time – unlikely that it can be done in quadratic time
  - 2-approximation in $O(n^{2.69})$ time
  - 8-approximation in $O(n^2)$ time (assuming input is distance matrix)

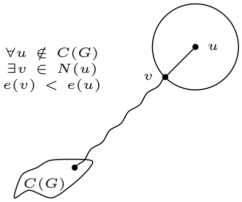
Prevalence of δ-hyperbolic graphs

- Any graph is δ-hyperbolic for some $\delta \leq diam(G)/2$
- Many special graph classes have a constant hyperbolicity. For example:
  - Interval graphs
  - Chordal graphs
  - Dually-chordal graphs
  - AT-free graphs
  - Weakly-chordal graphs
- Many real-world networks have a small hyperbolicity. For example:
  - biological networks
  - social networks
  - Internet application networks
  - collaboration networks, etc.

Every graph isometrically embeds into a hyperbolicity-preserving Helly graph.

Helly graphs (a.k.a. disk-Helly graphs or absolute retracts)

- A graph is *Helly* if every family of pairwise intersecting balls has a nonempty common intersection (i.e., satisfy the Helly property)
- Recognizable in polynomial time $O(n^4)$, $O(n^2 m)$
- Properties of Helly graphs
  - Unbounded tree-width
  - Unbounded clique-width
  - Cannot be characterized by some forbidden structure as they do not exclude any fixed minor
  - **Unimodal eccentricity function: every vertex of locally minimum eccentricity is a central vertex**



Importance of Helly graphs

- Broad generalization of dually chordal graphs, which contain all interval graphs, directed path graphs, and strongly chordal graphs
- Every graph isometrically embeds into a minimal Helly graph, called the injective hull, or tight span
- Discrete equivalent of hyperconvex metric spaces

- The diameter, radius, and a central vertex can be found in linear time for dually chordal graphs (Helly graphs in which the intersection graph of balls is chordal).
- It was open until recently whether there are truly subquadratic algorithms for these problems on general Helly graphs.
  - Randomized algorithms to compute the radius and diameter [1] and find a central vertex [2], running with high probability in $\tilde{\mathcal{O}}(m\sqrt{n})$ time (i.e., subquadratic in $n+m$)
  - Randomized linear-time algorithm for computing all eccentricities in C_4-free Helly graphs [1]
- We improve these results by presenting a deterministic $\mathcal{O}(m\sqrt{n})$ algorithm which computes not only the radius and diameter but also all vertex eccentricities in a Helly graph

[1] (Ducoffe and Dragan, 2021)
[2] (Ducoffe, 2020)

Our contribution to Helly graphs

- A deterministic $\mathcal{O}(m\sqrt{n})$ algorithm which computes radius, diameter, all vertex eccentricities
- A parameterized algorithm to compute
  - the radius and a central vertex in $O(\delta m)$ time
  - all vertex eccentricities in $O(\delta^2 m \log \delta)$ time – if δ is unknown
  - all vertex eccentricities in $O(\delta^2 m)$ time – if δ (or a constant approximation of it) is known
- We rely on new structural properties obtained for this class of graphs

Characterization of Helly graphs

Let $e_M(v) = \max_{u \in M} d_G(u,v)$, $diam_M(G) = \max_{v \in M} e_M(v)$, $rad_M(G) = \min_{v \in V} e_M(v)$, $C_M(G) = \{v \in V : e_M(v) = rad_M(G)\}$.

Theorem

For a graph G the following statements are equivalent:

(1) *G is Helly;*

(2) *the eccentricity function $e_M(\cdot)$ is unimodal for every set $M \subseteq V$;*

(3) *$e_M(v) = d_G(v, C_M(G)) + rad_M(G)$ holds for every set $M \subseteq V$ and every vertex $v \in V$;*

(4) *$2rad_M(G) - 1 \leq diam_M(G) \leq 2rad_M(G)$ holds for every set $M \subseteq V$;*

(5) *$rad_M(G) = \lfloor \frac{diam_M(G)+1}{2} \rfloor$ holds for every set $M \subseteq V$.*

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Rephrasing (1) $\iff$ (5):

Corollary

For every graph $G = (V, E)$, the family of all balls $\{N^r_G[v] : v \in V, r \in \mathbb{N}\}$ of G has the Helly property if and only if the family of k-neighborhoods $\{N^k_G[v] : v \in V\}$ of G has the Helly property for every natural number k.

Characterization of Helly graphs

Theorem

For every Helly graph G, a constant bound on one parameter from $\{\delta(G), \gamma(G), \beta(G), \kappa(G)\}$ implies a constant bound on all others.

- $\delta(G)$: the smallest half-integer $\delta \geq 0$ such that G is δ-hyperbolic.
- $\gamma(G)$: the largest integer $\gamma \geq 0$ such that G has a $(\gamma \times \gamma)$ rectilinear grid as an isometric subgraph.
- $\beta(G)$: the smallest integer $\beta \geq 0$ such that all balls in G are β-pseudoconvex. A vertex set S is β-*pseudoconvex* [3] if, for every vertices $x, y \in S$, any vertex $z \in I_G(x, y) \setminus S$ satisfies $\min\{d_G(z, x), d_G(z, y)\} \leq \beta$.
- $\kappa(G)$: the smallest integer $\kappa \geq 0$ such that $diam(C_M(G)) \leq \kappa$ for every set $M \subseteq V$.

Corollary

The hyperbolicity of an n-vertex Helly graph G is at most $\sqrt{n} + 1$.

Corollary

For any n-vertex Helly graph G, we have $diam(C(G)) \leq 2\sqrt{n} + 3$.

[3] Dragan and Guarnera, 2020

Compute a central vertex and radius of a (δ-hyperbolic) Helly graph

1. Find a vertex c with eccentricity $e(c) \leq rad(G) + 2\delta$ in $O(\delta m)$ time [4]
2. Certify that either c is a central vertex or find a neighbor v of c with $e(v) < e(c)$ in $O(m)$ time [5].
3. Repeat the previous step at most 2δ times to descend from c to a central vertex c^*.

Lemma

If G is a Helly graph, then one can compute a central vertex and the radius of G in $\mathcal{O}(\delta m)$ time, where δ is the hyperbolicity of G.

Corollary

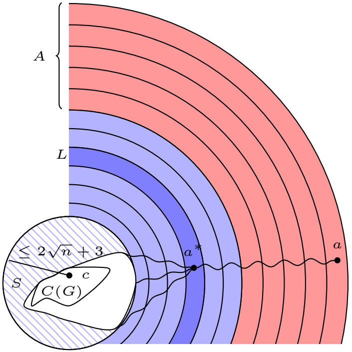
For any Helly graph G, a central vertex and the radius of G can be computed in $\mathcal{O}(m\sqrt{n})$ time.

[4] (Chepoi et al., 2008; Dragan and Guarnera, 2020; Dragan, Habib, and Viennot, 2018)
[5] (Ducoffe, 2020)

Compute all vertex eccentricities in a Helly graph in $\mathcal{O}(m\sqrt{n})$ time

Recall that for every $v \in V$, $e(v) = d(v, C(G)) + rad(G)$. **Goal: find the center** $C(G)$.

- Find a central vertex c in $\mathcal{O}(m\sqrt{n})$ time.
- Take set S of vertices within $2\sqrt{n} + 3$ of c. We know $C(G) \subseteq S$.
- Eliminate from S each $u \in S$ which is non-central
    - identify if there are vertices v far from u such that $d(u, v) > rad(G)$
    - consider BFS layering from S
    - potential vertices farthest from u belong in farthest layers from S; let A be the union of all such 'far' layers
    - there is a layer L of size $O(\sqrt{n})$ which contains, for each $a \in A$, a *gate vertex* a^* which lays upon a shortest path from a to every vertex of S within $rad(G)$ of a.
    - we need only focus on distances to gate vertices



Compute all vertex eccentricities in a δ-hyperbolic Helly graph

Theorem

If G is an m-edge Helly graph of hyperbolicity δ, then the eccentricity of all vertices of G can be computed in $\mathcal{O}(\delta^2 m \log \delta)$ time. The algorithm does not need to know the value of δ in order to work correctly. If δ (or a constant approximation of it) is known, then the running time is $\mathcal{O}(\delta^2 m)$.

We can compute in linear time all vertex eccentricities in a Helly graph if its hyperbolicity δ is a constant, e.g., dually chordal graphs ($\delta \leq 1$) and strongly chordal graphs ($\delta \leq 1$).

Conclusion and future work

Conclusion:

- We present the first truly subquadratic deterministic algorithm for computing all eccentricities in a Helly graph
- For Helly graphs with constant hyperbolicity, we gave a linear time algorithm
- We obtain new structural properties of Helly graphs

Next steps:

- What other algorithmic graph problems can efficiently be solved on Helly graphs utilizing $\delta \leq \sqrt{n} + 1$?
- The Helly property for balls of equal radii implies the Helly property for balls with variable radii. Does a similar result hold for all (discrete) metric spaces?
- Helly graphs were recently generalized to graphs with a bounded Helly-gap. [6] Can our results be extended to all graphs with bounded Helly-gap?

[6] (Dragan and Guarnera, 2021)