Exam 2 Review Notes

The following is a collection of notes from the Exam 2 Review session. **It does not constitute a comprehensive list of topics and problems required to know for the exam.** Its purpose is to provide additional examples on some important concepts from Chapters 2 and 5.

1 Logic and English Translation

It's important here to know some important sets in math. Z is the set of integers $\{...-2,-1,0,1,2,...\}$. N is the set of natural numbers $\{0,1,2,3,...\}$. R is the set of real numbers (anything on the real number line).

1. $\forall x \in N \quad x \in Z$

   Solution: *All natural numbers are integers.*

2. Any real number divided by two is a real number.

   Solution: $\forall x \in R \quad \frac{x}{2} \in R$

2 Set Operations and Definitions

In the following examples, let $A = \{1,2,3\}$, $B = \{1,2,3,4,5,6\}$, $C = \{5,6\}$, and the universal set $U = \{1,2,3,4,5,6,7,8,9\}$.

1. $A \cup C$

   Solution: $\{1,2,3,5,6\}$

2. $A \cap B$

   Solution: $\{1,2,3\}$

3. $\overline{B}$

   Solution: $\{7,8,9\}$

4. $A - B$

   Solution: $\emptyset$

5. $B - A$

   Solution: $\{4,5,6\}$

6. $|C|$

   Solution: *2*

7. $A \times C$

   Solution: $\{(1,5),(1,6),(2,5),(2,6),(3,5),(3,6)\}$

8. $\mathcal{P}(A)$

   Solution: $\{\emptyset,\{1\},\{2\},\{3\},\{1,2\},\{1,3\},\{2,3\},\{1,2,3\}\}$

9. Give a recursive definition of the set of all even positive integers (i.e., $\{2,4,6,8,10,...\}$).

Solution: *$2 \in S$, and if $x \in S$ then $x + 2 \in S$.*

3 Venn Diagrams

Draw a Venn Diagram to show the relationship between the sets ($\subseteq$, $=$, $\supseteq$, no relation)...

1. ...for the set $(A \cap C) \cup (B \cap C)$ and the set $A \cap B \cap C$
Solution:

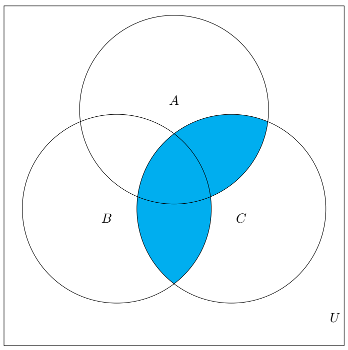

Figure 1: $(A \cap C) \cup (B \cap C)$.

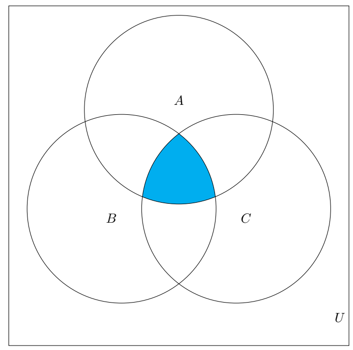

Figure 2: $A \cap B \cap C$.

As you can see from the figures above, $A \cap B \cap C \subseteq (A \cap C) \cup (B \cap C)$.

4 Set Equality

We can prove that two sets are equal using one of three methods: a membership table, set builder notation and propositional logic, or a containment proof.

1. Prove the absorption law for sets (that $A \cup (A \cap B) = A$) using a containment proof.

   Solution: *Suppose $x \in A \cup (A \cap B)$. Then $x \in A$ or $x \in A \cap B$ by the definition of union. In the former case, we have $x \in A$, and in the latter case we have $x \in A$ and $x \in B$ by the definition of intersection; thus in any event, $x \in A$, so we have proved that the left-hand side is a subset of the right-hand side. Conversely, let $x \in A$. Then by the definition of union, $x \in A \cup (A \cap B)$ as well. Thus we have shown that the right-hand side is a subset of the left-hand side. Q.E.D.*

2. Prove the absorption law for sets (that $A \cup (A \cap B) = A$) using set-builder notation.

   Solution:

$$\begin{aligned}
A \cup (A \cap B) &= \{x | x \in A \vee x \in (A \cap B)\} && \textit{(by definition)} \\
&= \{x \in A \vee (x \in A \wedge x \in B)\} && \textit{(by definition of intersection)} \\
&= \{x | x \in A\} && \textit{(by absorption law for propositions)} \\
&= A && \textit{(by notation)}
\end{aligned}$$

3. Prove the absorption law for sets (that $A \cup (A \cap B) = A$) using a membership table.

Solution:

A	B	$A \cap B$	$A \cup (A \cap B)$
0	*0*	*0*	*0*
0	*1*	*0*	*0*
1	*0*	*0*	*1*
1	*1*	*1*	*1*

Since A and $A \cup (A \cap B)$ have the same values in each instance, they represent the same set.

5 Sequences

1. What is the value of $\sum_{i=3}^{4} i^2$?

   Solution: $3^2 + 4^2 = 9 + 16 = 25$

2. Let $\{a_n\} = 3 + n$. What is the value of a_0, a_1, and a_2?

   Solution: $a_0 = 3$, $a_1 = 4$, *and* $a_2 = 5$.

6 Methods of Iteration

1. Use forward or backward substitute to conjecture a closed formula which describes the n^{th} term of the sequence $a_n = a_{n-1} - n$ where $a_0 = 4$.

   Solution:

$$\begin{aligned}
&a_0 = 4 \\
&a_1 = a_0 - 1 = 4 - 1 \\
&a_2 = a_1 - 2 = 4 - 1 - 2 \\
&a_3 = a_2 - 3 = 4 - 1 - 2 - 3 \\
&a_4 = a_3 - 4 = 4 - (1 + 2 + 3 + 4) \\
&a_n = 4 - (1 + 2 + 3 + 4..... + n) = 4 - \tfrac{n \cdot (n+1)}{2}
\end{aligned}$$

7 Functions

1. If $f : R \to Z$ and $f(x) = \lfloor x \rfloor$, what is the domain and what is the range of the function?

   Solution: *The domain is R, and the range is Z.*

2. Determine whether the function $f : Z \to Z$ is surjective (onto) if $f(x) = x + 1$.

   Solution: *Yes, it's surjective. It's possible to reach every integer in the codomain by plugging in some integer.*

3. Determine whether the function $f : Z \to Z$ is injective (one-to-one) if $f(x) = x^2$.

Solution: *No, it's not injective. For example,* $f(-1) = f(1) = 1$.

4. Find the inverse for the function $f : Z \rightarrow Z$, where $f(x) = x + 1$, if the inverse exists.
   Solution: *The inverse exists because* f *is bijective (both injective and surjective). The inverse is* $f^{-1}(y) = y - 1$.
5. Give a recursive definition for the factorial function.
   Solution: $f(0) = 1$ *and* $f(n) = (n) \cdot f(n-1)$.
6. Given a function recursively defined by $f(0) = 1$ and $f(n) = \frac{1}{f(n-1)}$, find $f(5)$.
   Solution: $f(1) = 1$, $f(2) = 1$, $f(3) = 1$, $f(4) = 1$, *so therefore* $f(5) = 1$.
7. If $f(x) = 3x$ and $g(x) = x^2$, where $f : R \rightarrow R$ and $g : R \rightarrow R$ what is $f \circ g$?
   Solution: $(f \circ g)(x) = f(g(x)) = 3(x^2) = 3x^2$.
8. If $f(x) = 3x$ and $g(x) = x^2$, where $f : R \rightarrow R$ and $g : R \rightarrow R$ what is $g \circ f$?
   Solution: $(g \circ f)(x) = g(f(x)) = (3x)^2 = 9x^2$.
9. If $f : Z \rightarrow Z$ and $f(x) = \lfloor \frac{x}{2} \rfloor + \lceil \frac{3x}{2} \rceil + (x \bmod 2) + x!$, what is $f(3)$?
   Solution: $f(3) = \lfloor \frac{3}{2} \rfloor + \lceil \frac{3 \cdot 3}{2} \rceil + (3 \ mod\ 2) + 3! = 1 + 5 + 1 + 6 = 13$

8 Mathematical Induction

For any proof by mathematical induction, always clearly label the basis step and inductive step, and clearly state the assumptions in the inductive step.

To prove a statement of the form $\forall n \geq b \quad P(n)$, the basis step proves that $P(b)$ is true, and the inductive step proves that $P(k) \rightarrow P(k+1)$ for $k \geq b$. Use a direct proof to prove the inductive step, i.e., assume that $P(k)$ is true and use this information to show that $P(k+1)$ is also true.

In order to determine what b is from the problem description, it is helpful to know what it means for an integer n to be positive ($n > 0$) vs. non-negative ($n \geq 0$).

1. Let $P(n)$ be the statement that $n! < n^n$. Prove using mathematical induction that $P(n)$ is true for any integer $n \geq 2$.
   Solution:

   Basis Step. $P(2)$ *is true, since* $2! = 2 < 2^2 = 4$.

   Inductive Step. *Assume* $P(k)$ *is true - that* $k! < k^k$ *for* $k \geq 2$. *We'll show* $P(k+1)$, *that* $(k+1)! < (k+1)^{k+1}$. *We know that* $(k+1)! = k! \cdot (k+1) < k^k \cdot (k+1) < (k+1)^k \cdot (k+1) = (k+1)^{k+1}$. *Q.E.D.*
2. Prove using mathematical induction that 3 divides $n^3 + 2n$ whenever n is a positive integer.
   Solution:

   Basis Step. $P(1)$ *is true since* $1^3 + 2(1) = 3$, *which is evenly divisible by 3.*

   Inductive Step. *Assume* $P(k)$ *is true - that* $k^3 + 2k$ *is evenly divisible by 3. We'll show that* $P(k+1)$ *is true - that* $(k+1)^3 + 2(k+1)$ *is evenly divisible by 3. We know that*

   $$\begin{aligned} (k+1)^3 + 2(k+1) &= k^3 + 3k^2 + 3k + 1 + 2(k+1) \\ &= k^3 + 3k^2 + 3k + 2k + 3 \\ &= (k^3 + 2k) + 3(k^2 + k + 1) \end{aligned}$$

   Since $(k^3 + 2k)$ *is evenly divisible by 3 by the inductive hypothesis, and* $3(k^2 + k + 1)$ *is evenly divisible by 3, then* $(k+1)^3 + 2(k+1)$ *is evenly divisible by 3. Q.E.D.*

9 Structural Induction

1. The set of leaves and the set of internal vertices of a full binary tree can be defined recursively.

   Basis step: The root r is a leaf of the full binary tree with exactly one vertex r. This tree has no internal vertices.

   Recursive step: The set of leaves of the tree $T = T_1 \cdot T_2$ is the union of the sets of leaves of T_1 and of T_2. The internal vertices of T are the root r of T and the union of the set of internal vertices of T_1 and the set of internal vertices of T_2.

   Use structural induction to show that $l(t)$, which is the number of leaves of a full binary tree T, is 1 more than $i(T)$, which is the number of internal vertices of T.

   Solution:

   Basis Step. *Consider a single vertex r, which has one leaf and no internal vertices. Therefore $l(r) = 1 = i(r) + 1 = 0 + 1$.*

   Recursive Step. *Assume this property holds for T_1 and for T_2, and consider the larger tree T with a new root whose children are T_1 and T_2. We know that*

$$\begin{aligned} l(T) &= l(T_1) + l(T_2) \\ &= i(T_1) + 1 + i(T_2) + 1 && \textit{(inductive hypothesis)} \\ &= i(T) + 1 && \textit{(from definition of i(T))} \end{aligned}$$

   Q.E.D.