Expected Value

Section 7.4 (partially)

Section Summary

- Expected Value
- Linearity of Expectations
- Independent Random Variables

Expected Value

Definition: The *expected value* (or *expectation* or *mean*) of a random variable $X(s)$ on the sample space S is equal to

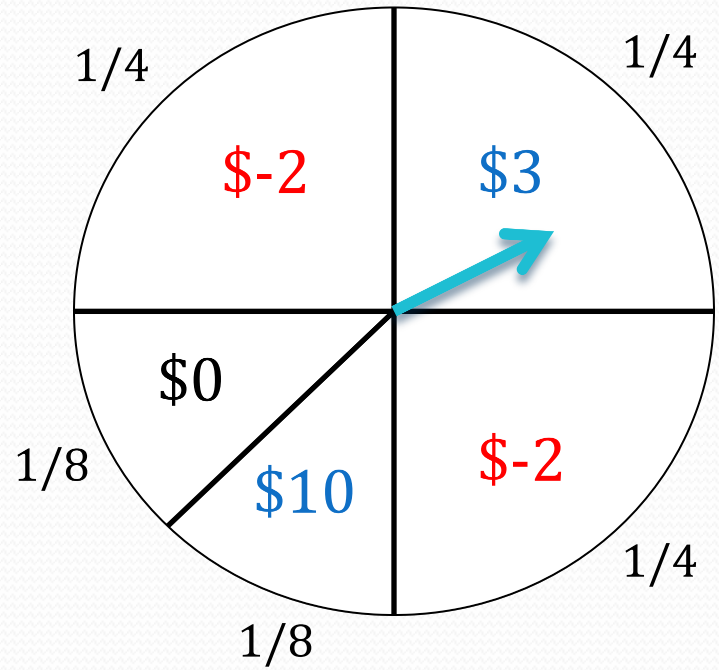
$$E(X) = \sum_{s \in S} p(s) X(s)$$

Example-Expected Value of a Die: Let X be the number that comes up when a fair die is rolled. What is the expected value of X?

Solution: The random variable X takes the values 1, 2, 3, 4, 5, or 6. Each has probability 1/6. It follows that

$$E(X) = \frac{1}{6} \cdot 1 + \frac{1}{6} \cdot 2 + \cdots \frac{1}{6} \cdot 6 = \frac{21}{6} = \frac{7}{2}.$$

Expected Value



Example: How much money can you expect to win (or lose!) playing this spin-the-wheel game?

Solution: Possible outcomes are

- Lose \$2, with 1/2 probability
- Break even (\$0) with 1/8 probability
- Win \$3 with 1/4 probability
- Win \$10 with 1/8 probability

$$E(X) = -2\cdot(1/2) + 0\cdot(1/8) + 3\cdot(1/4) + 10\cdot(1/8)$$
$$= -1 + 3/4 + 10/8 \quad = -8/8 + 6/8 + 10/8 \quad = 1$$

Expect to win, on average, \$1 per game.

Expected Value

Theorem 1: If X is a random variable and $p(X = r)$ is the probability that X = r, so that $p(X = r) = \sum_{s \in S, X(s)=r} p(s)$, then

$$E(X) = \sum_{r \in X(S)} p(X = r) r.$$

see the text for the proof

Expected Value

Example: What is the expected sum of the numbers that appear when two fair dice are rolled?

Solution: Let X be the sum of the numbers, with range 2-12

- p(X=2) = p(X=12) = 1/36
- p(X=3) = p(X=11) = 2/36 = 1/18
- p(X=4) = p(X=10) = 3/36 = 1/12
- p(X=5) = p(X=9) = 4/36 = 1/9
- p(X=6) = p(X=8) = 5/36,
- p(X=7) = 6/36 = 1/6.

$$\begin{aligned} E(X) &= 2 \cdot \frac{1}{36} + 3 \cdot \frac{1}{18} + 4 \cdot \frac{1}{12} + 5 \cdot \frac{1}{9} + 6 \cdot \frac{5}{36} + 7 \cdot \frac{1}{6} \\ &\quad + 8 \cdot \frac{5}{36} + 9 \cdot \frac{1}{9} + 10 \cdot \frac{1}{12} + 11 \cdot \frac{1}{18} + 12 \cdot \frac{1}{36} \\ &= 7. \end{aligned}$$

Expected Value

Theorem 2: The expected number of successes when n mutually independent Bernoulli trials are performed is np, where p is the probability of success on each trial.

see the text for the proof

Example: What is the expected number of heads that come up when a fair coin is flipped 5 times?

Solution: By theorem 2, with $n=5$ and $p=1/2$, the expected number of heads is $5 \cdot (1/2) = 2.5$

Linearity of Expectations

The following theorem tells us that expected values are linear. For example, the expected value of the sum of random variables is the sum of their expected values.

Theorem 3: If X_i, $i = 1, 2, \ldots, n$ with n a positive integer, are random variables on S, and if a and b are real numbers, then

(i) $E(X_1 + X_2 + \ldots. + X_n) = E(X_1) + E(X_2) + \ldots. + E(X_n)$

(ii) $E(aX + b) = aE(X) + b$.

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see the text for the proof

Linearity of Expectations

Example: What is the expected value of the sum of the numbers that appear when a pair of fair dice is rolled?

Solution: Let X_1 be the number on the first die and X_2 be the number on the second die.

- $E(X_1) = 7/2$
- $E(X_2) = 7/2$

Then, $E(X_1+X_2) = 7/2 + 7/2 = 7$

Independent Random Variables

Definition 3: The random variables X and Y on a sample space S are independent if

$$p(X = r_1 \text{ and } Y = r_2) = p(X = r_1) \cdot p(Y = r_2).$$

Theorem 5: If X and Y are independent variables on a sample space S, then $E(X \cdot Y) = E(X) \cdot E(Y)$.

see text for the proof

Independent Random Variables

Example: Suppose we throw independent, fair dice and multiply the numbers that come up. What is the expected value of this product?

Solution: Let X and Y be the numbers shown on the first and second dice. Their expected product is then

$$E(X \cdot Y) = E(X) \cdot E(Y)$$
$$= 7/2 \cdot 7/2$$
$$= 49/4$$

Independent Random Variables

Example: Let X and Y be random variables that count the number of heads and tails when a fair coin is flipped twice. What is $E(X \cdot Y)$?

Solution: X and Y are dependent.

	2 heads/tails	1 head/tail	0 heads/tails
$E(X) =$	$2 \cdot (1/4)$	$+ 1 \cdot (1/2)$	$+ 0 \cdot (1/4) = 1$
$E(Y) =$	$2 \cdot (1/4)$	$+ 1 \cdot (1/2)$	$+ 0 \cdot (1/4) = 1$

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	1 of each	2 of one	
$E(X \cdot Y) =$	$1 \cdot (1/2)$	$+ 0 \cdot (1/2) = ½$	$\neq E(X) \cdot E(Y)$