# Nested Quantifiers

Section 1.5

# Section Summary

- Nested Quantifiers
- Order of Quantifiers
- Translating from Nested Quantifiers into English
- Translating Mathematical Statements into Statements involving Nested Quantifiers.
- Translating English Sentences into Logical Expressions.
- Negating Nested Quantifiers.

# Nested Quantifiers

- Nested quantifiers are often necessary to express the meaning of sentences in English as well as important concepts in computer science and mathematics.

  **Example**: "Every real number has an inverse" is

  $\forall x\, \exists y (x + y = 0)$

  where the domains of x and y are the real numbers.

- We can also think of nested propositional functions:

  $\forall x\, \exists y (x + y = 0)$ can be viewed as $\forall x\, Q(x)$ where $Q(x)$ is $\exists y\, P(x, y)$ where $P(x, y)$ is $(x + y = 0)$

# Thinking of Nested Quantification

- Nested Loops
  - To see if $\forall x \forall y P(x,y)$ is true, loop through the values of $x$ :
    - At each step, loop through the values for $y$.
    - If for some pair of $x$ and $y$, $P(x,y)$ is false, then $\forall x\, \forall y P(x,y)$ is false and both the outer and inner loop terminate.

    $\forall x\, \forall y\, P(x,y)$ is true if the outer loop ends after stepping through each $x$.
  - To see if $\forall x\, \exists y P(x,y)$ is true, loop through the values of $x$:
    - At each step, loop through the values for $y$.
    - The inner loop ends when a pair $x$ and $y$ is found such that $P(x, y)$ is true.
    - If no $y$ is found such that $P(x, y)$ is true the outer loop terminates as $\forall x\, \exists y P(x,y)$ has been shown to be false.

    $\forall x\, \exists y\, P(x,y)$ is true if the outer loop ends after stepping through each $x$.
- If the domains of the variables are infinite, then this process can not actually be carried out.

# Order of Quantifiers

**Examples**:

1. Let $P(x,y)$ be the statement "$x + y = y + x$." Assume that $U$ is the real numbers. Then $\forall x \, \forall y P(x,y)$ and $\forall y \, \forall x P(x,y)$ have the same truth value.
2. Let $Q(x,y)$ be the statement "$x + y = 0$." Assume that $U$ is the real numbers. Then $\forall x \, \exists y Q(x,y)$ is true, but $\exists y \, \forall x Q(x,y)$ is false.

# Questions on Order of Quantifiers

**Example 1**: Let $U$ be the real numbers,

Define $P(x,y) : x \cdot y = 0$

What is the truth value of the following:

1. $\forall x \forall y P(x,y)$

   **Answer:** False

2. $\forall x \exists y P(x,y)$

   **Answer:** True

3. $\exists x \forall y \, P(x,y)$

   **Answer:** True

4. $\exists x \, \exists \, y \, P(x,y)$

   **Answer:** True

# Questions on Order of Quantifiers

**Example 2**: Let $U$ be the real numbers,

Define $P(x,y) : x / y = 1$

What is the truth value of the following:

1. $\forall x \forall y P(x,y)$

   **Answer:** False

2. $\forall x \exists y P(x,y)$

   **Answer:** True

3. $\exists x \forall y\ P(x,y)$

   **Answer:** False

4. $\exists x\ \exists\ y\ P(x,y)$

   **Answer:** True

# Quantifications of Two Variables

| Statement | When True? | When False |
|---|---|---|
| $\forall x \forall y P(x,y)$<br>$\forall y \forall x P(x,y)$ | *P*(*x*,*y*) is true for every pair *x*,*y*. | There is a pair *x*, *y* for which *P*(*x*,*y*) is false. |
| $\forall x \exists y P(x,y)$ | For every *x* there is a *y* for which *P*(*x*,*y*) is true. | There is an x such that *P*(*x*,*y*) is false for every *y*. |
| $\exists x \forall y P(x,y)$ | There is an *x* for which *P*(*x*,*y*) is true for every *y*. | For every *x* there is a y for which *P*(x,y) is false. |
| $\exists x \exists y P(x,y)$<br>$\exists y \exists x P(x,y)$ | There is a pair *x*, *y* for which *P*(*x*,*y*) is true. | *P*(x,y) is false for every pair *x*,*y* |

# Translating Nested Quantifiers into English

**Example 1**: Translate the statement

$$\forall x \ (C(x) \vee \exists y \ (C(y) \wedge F(x, y)))$$

where C(x) is "*x* has a computer," and *F*(*x*,*y*) is "*x* and *y* are friends," and the domain for both *x* and *y* consists of all students in your school.

**Solution**: Every student in your school has a computer or has a friend who has a computer.

**Example 2**: Translate the statement

$$\exists x \forall y \ \forall z \ ((F(x, y) \wedge F(x,z) \wedge (y \neq z)) \rightarrow \neg F(y,z))$$

**Solution**: There is a student none of whose friends are also friends with each other.

# Translating Mathematical Statements into Predicate Logic

**Example** : Translate "The sum of two positive integers is always positive" into a logical expression.

**Solution**:

1. Rewrite the statement to make the implied quantifiers and domains explicit:

   "For every two integers, if these integers are both positive, then the sum of these integers is positive."

2. Introduce the variables *x* and *y*, and specify the domain, to obtain:

   "For all positive integers *x* and *y*, *x* + *y* is positive."

3. The result is:

$$\forall x \, \forall y \, ((x > 0) \wedge (y > 0) \rightarrow (x + y > 0))$$

where the domain of both variables consists of all integers

# Translating English into Logical Expressions Example

**Example**: Use quantifiers to express the statement "There is a woman who has taken a flight on every airline in the world."

**Solution**:

1. Let $P(w,f)$ be "$w$ has taken $f$" and $Q(f,a)$ be "$f$ is a flight on $a$."
2. The domain of $w$ is all women, the domain of $f$ is all flights, and the domain of $a$ is all airlines.
3. Then the statement can be expressed as:

$$\exists w\, \forall a\, \exists f\, (P(w,f) \wedge Q(f,a))$$

# Questions on Translation from English

Choose the obvious predicates and express in predicate logic.

**Example 1**: "Brothers are siblings."

**Solution**: $\forall x\ \forall y\ (B(x,y) \rightarrow S(x,y))$

**Example 2**: "Siblinghood is symmetric."

**Solution**: $\forall x\ \forall y\ (S(x,y) \rightarrow S(y,x))$

**Example 3**: "Everybody loves somebody."

**Solution**: $\forall x\ \exists y\ L(x,y)$

**Example 4**: "There is someone who is loved by everyone."

**Solution**: $\exists y\ \forall x\ L(x,y)$

**Example 5**: "There is someone who loves someone."

**Solution**: $\exists x\ \exists y\ L(x,y)$

**Example 6**: "Everyone loves himself"

**Solution**: $\forall x\ L(x,x)$

# Negating Nested Quantifiers

**Example 1**: Express the negation of the statement $\forall x \exists y (xy = 1)$ so that no negation precedes a quantifier.

**Solution**: Use De Morgan's Laws to move the negation as far inwards as possible.

1. $\neg \forall x \exists y (xy = 1)$
2. $\exists x \neg \exists y (xy = 1)$ by De Morgan's for $\forall$
3. $\exists x \forall y \neg (xy = 1)$ by De Morgan's for $\exists$
4. $\exists x \forall y (xy \neq 1)$

# Negating Nested Quantifiers

**Example 1**: Recall the logical expression developed three slides back:

$$\exists w \forall a \exists f\ (P(w,f) \wedge Q(f,a))$$

**Part 1**: Use quantifiers to express the statement that "There does not exist a woman who has taken a flight on every airline in the world."

**Solution**: $\neg\exists w \forall a \exists f\ (P(w,f) \wedge Q(f,a))$

**Part 2**: Now use De Morgan's Laws to move the negation as far inwards as possible.

**Solution**:

1. $\neg\exists w \forall a \exists f\ (P(w,f) \wedge Q(f,a))$
2. $\forall w \neg \forall a \exists f\ (P(w,f) \wedge Q(f,a))$ by De Morgan's for $\exists$
3. $\forall w \exists a \neg \exists f\ (P(w,f) \wedge Q(f,a))$ by De Morgan's for $\forall$
4. $\forall w \exists a \forall f \neg (P(w,f) \wedge Q(f,a))$ by De Morgan's for $\exists$
5. $\forall w \exists a \forall f (\neg P(w,f) \vee \neg Q(f,a))$ by De Morgan's for $\wedge$.

**Part 3**: Can you translate the result back into English?

**Solution**:

"For every woman there is an airline such that for all flights, this woman has not taken that flight or that flight is not on this airline"