

Expected Value

Section 7.4 (partially)

Section Summary

- Expected Value
- Linearity of Expectations
- Independent Random Variables

Expected Value

Definition: The *expected value* (or *expectation* or *mean*) of a random variable $X(s)$ on the sample space S is equal to

$$E(X) = \sum_{s \in S} p(s)X(s)$$

Example-Expected Value of a Die: Let X be the number that comes up when a fair die is rolled. What is the *expected value* of X ?

Solution: The random variable X takes the values 1, 2, 3, 4, 5, or 6. Each has probability $1/6$. It follows that

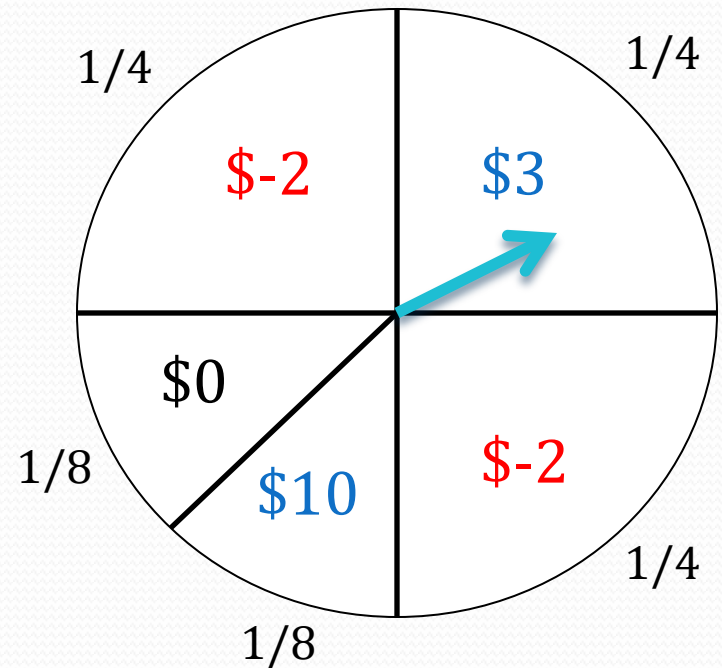
$$E(X) = \frac{1}{6} \cdot 1 + \frac{1}{6} \cdot 2 + \cdots + \frac{1}{6} \cdot 6 = \frac{21}{6} = \frac{7}{2}.$$

Expected Value

Example: How much money can you **expect** to win (or lose!) playing this spin-the-wheel game?

Solution: Possible outcomes are

- Lose \$2, with 1/2 probability
- Break even (\$0) with 1/8 probability
- Win \$3 with 1/4 probability
- Win \$10 with 1/8 probability



$$\begin{aligned} E(X) &= -2 \cdot (1/2) + 0 \cdot (1/8) + 3 \cdot (1/4) + 10 \cdot (1/8) \\ &= -1 + 3/4 + 10/8 = -8/8 + 6/8 + 10/8 = 1 \end{aligned}$$

Expect to win, on average, \$1 per game.

Expected Value

Theorem 1: If X is a random variable and $p(X = r)$ is the probability that $X = r$, so that $p(X = r) = \sum_{s \in S, X(s)=r} p(s)$, then

$$E(X) = \sum_{r \in X(S)} p(X = r)r.$$

see the text for the proof

Expected Value

Example: What is the expected sum of the numbers that appear when two fair dice are rolled?

Solution: Let X be the sum of the numbers, with range 2-12

- $p(X=2) = p(X=12) = 1/36$
- $p(X=3) = p(X=11) = 2/36 = 1/18$
- $p(X=4) = p(X=10) = 3/36 = 1/12$
- $p(X=5) = p(X=9) = 4/36 = 1/9$
- $p(X=6) = p(X=8) = 5/36,$
- $p(X=7) = 6/36 = 1/6.$

$$\begin{aligned} E(X) &= 2 \cdot \frac{1}{36} + 3 \cdot \frac{1}{18} + 4 \cdot \frac{1}{12} + 5 \cdot \frac{1}{9} + 6 \cdot \frac{5}{36} + 7 \cdot \frac{1}{6} \\ &\quad + 8 \cdot \frac{5}{36} + 9 \cdot \frac{1}{9} + 10 \cdot \frac{1}{12} + 11 \cdot \frac{1}{18} + 12 \cdot \frac{1}{36} \\ &= 7. \end{aligned}$$

Expected Value

Theorem 2: The **expected number of successes** when n mutually independent Bernoulli trials are performed is np , where p is the probability of success on each trial.

see the text for the proof

Example: What is the expected number of heads that come up when a fair coin is flipped 5 times?

Solution: By theorem 2, with $n=5$ and $p=1/2$, the expected number of heads is $5 \cdot (1/2) = 2.5$

Linearity of Expectations

The following theorem tells us that **expected values are linear**. For example, the expected value of the sum of random variables is the sum of their expected values.

Theorem 3: If X_i , $i = 1, 2, \dots, n$ with n a positive integer, are random variables on S , and if a and b are real numbers, then

$$(i) E(X_1 + X_2 + \dots + X_n) = E(X_1) + E(X_2) + \dots + E(X_n)$$

$$(ii) E(aX + b) = aE(X) + b.$$

see the text for the proof

Linearity of Expectations

Example: What is the expected value of the sum of the numbers that appear when a pair of fair dice is rolled?

Solution: Let X_1 be the number on the first die and X_2 be the number on the second die.

- $E(X_1) = 7/2$
- $E(X_2) = 7/2$

Then, $E(X_1 + X_2) = 7/2 + 7/2 = 7$

Independent Random Variables

Definition 3: The random variables X and Y on a sample space S are **independent** if

$$p(X = r_1 \text{ and } Y = r_2) = p(X = r_1) \cdot p(Y = r_2).$$

Theorem 5: If X and Y are independent variables on a sample space S , then $E(X \cdot Y) = E(X) \cdot E(Y)$.

see text for the proof

Independent Random Variables

Example: Suppose we throw **independent**, fair dice and multiply the numbers that come up. What is the expected value of this product?

Solution: Let X and Y be the numbers shown on the first and second dice. Their expected product is then

$$\begin{aligned} E(X \cdot Y) &= E(X) \cdot E(Y) \\ &= 7/2 \cdot 7/2 \\ &= 49/4 \end{aligned}$$

Dependent Random Variables

Example: Let X and Y be random variables that count the number of heads and tails when a fair coin is flipped twice. What is $E(X \cdot Y)$?

Solution: X and Y are **dependent**.

$$E(X) = 2 \cdot (1/4) + 1 \cdot (1/2) + 0 \cdot (1/4) = 1$$

$$E(Y) = 2 \cdot (1/4) + 1 \cdot (1/2) + 0 \cdot (1/4) = 1$$

$$E(X \cdot Y) = 1 \cdot (1/2) + 0 \cdot (1/2) = 1/2 \neq E(X) \cdot E(Y)$$

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