# Functions

Section 2.3

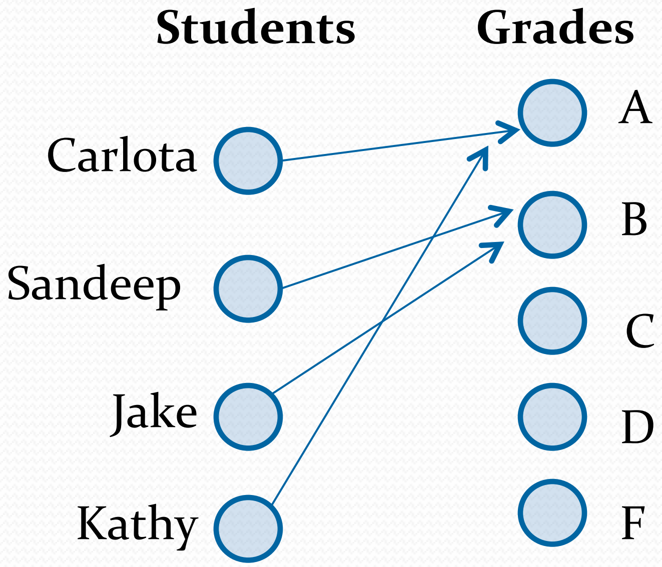
# Section Summary

- Definition of a Function.
  - Domain, Codomain
  - Image, Preimage
- Injection, Surjection, Bijection
- Inverse Function
- Function Composition
- Graphing Functions
- Floor, Ceiling, Factorial

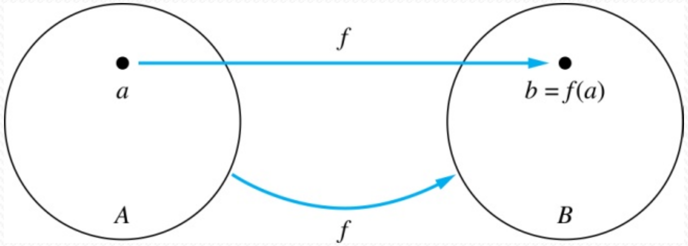
# Functions

**Definition**: Let $A$ and $B$ be nonempty sets. A *function* $f$ from $A$ to $B$, denoted $f: A \to B$ is an assignment of each element of $A$ to exactly one element of $B$. We write $f(a)=b$ if $b$ is the unique element of $B$ assigned by the function $f$ to the element $a$ of $A$.

- Functions are sometimes called *mappings* or *transformations*.



# Given a function $f$: $A \rightarrow B$




- We say $f$ *maps* $A$ to $B$ or
  $f$ is a *mapping* from $A$ to $B$.
- $A$ is called the *domain* of $f$.
- $B$ is called the *codomain* of $f$.
- If $f(a) = b$,
  - then $b$ is called the *image* of $a$ under $f$.
  - $a$ is called the *preimage* of $b$.
- The *range* of $f$ is the set of all images of points in **A** under $f$. We denote it by $f(A)$.
- Two functions are *equal* when they have the same domain, the same codomain, and map each element of the domain to the same element of the codomain.

# Representing Functions

Functions may be specified in different ways:

- An explicit statement of the assignment.

  Students and grades example.

- A formula.

  $f(x) = x + 1$

- A computer program.

  A C++ program that when given an integer $n$, produces the $n$th Fibonacci Number (covered in the next section and also in Ch. 5).

# Questions

$f(\mathrm{a}) = ?$ z

The image of d is ? z

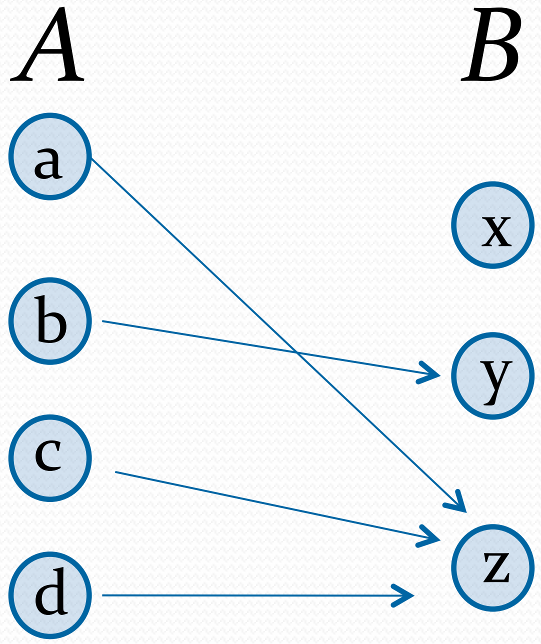
The domain of f is ? $A$

The codomain of f is ? $B$

The preimage of y is ? b

$f(A) = ?$ {y,z}

The preimage(s) of z is (are) ? {a,c,d}



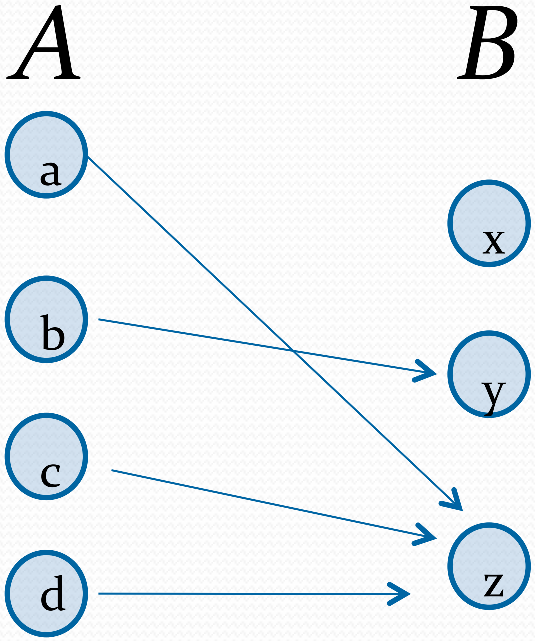
# Question on Functions and Sets

- If $f : A \to B$ and S is a subset of A, then

$$f(S) = \{f(s) | s \in S\}$$

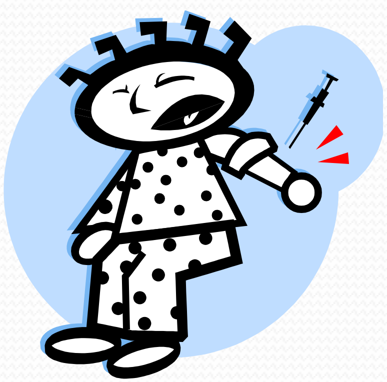
$f$\{a,b,c\} is ? \{y,z\}

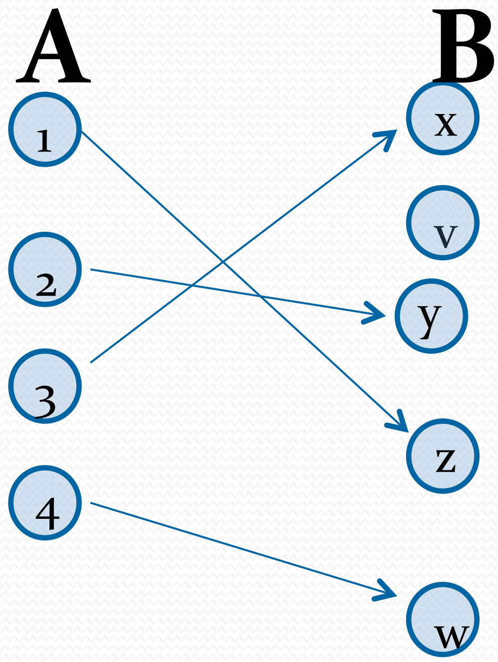
$f$\{c,d\} is ? \{z\}



# Injections

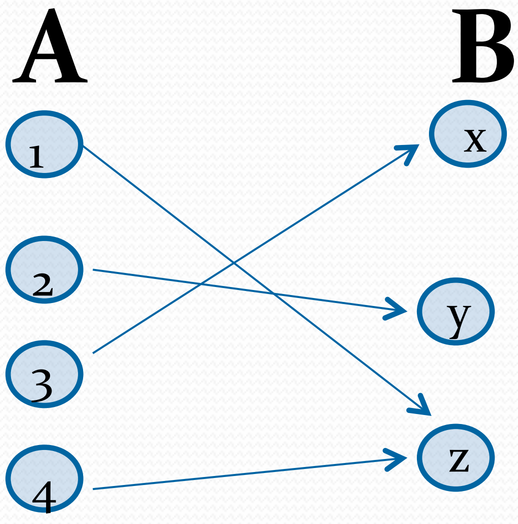
**Definition**: A function f is said to be *one-to-one*, or *injective*, if and only if $f(a) = f(b)$ implies that $a = b$ for all $a$ and $b$ in the domain of $f$. A function is said to be an *injection* if it is one-to-one.





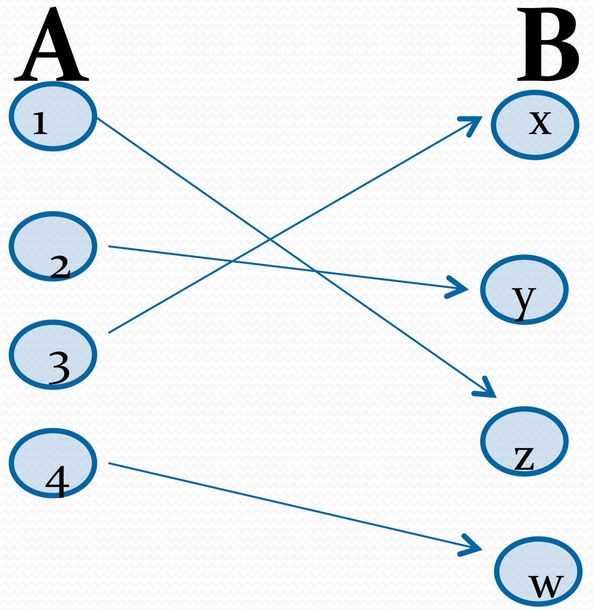
# Surjections

**Definition**: A function *f* from *A* to *B* is called *onto* or *surjective*, if and only if for every element $b \in B$ there is an element $a \in A$ with $f(a) = b$. A function *f* is called a *surjection* if it is onto.



# Bijections

**Definition**: A function f is a *one-to-one correspondence*, or a *bijection*, if it is both one-to-one and onto (surjective and injective).



# Showing that *f* is injective or surjective

Suppose that $f : A \to B$.

*To show that $f$ is injective* Show that if $f(x) = f(y)$ for arbitrary $x, y \in A$ with $x \neq y$, then $x = y$.

*To show that $f$ is not injective* Find particular elements $x, y \in A$ such that $x \neq y$ and $f(x) = f(y)$.

*To show that $f$ is surjective* Consider an arbitrary element $y \in B$ and find an element $x \in A$ such that $f(x) = y$.

*To show that $f$ is not surjective* Find a particular $y \in B$ such that $f(x) \neq y$ for all $x \in A$.

# Showing that *f* is injective or surjective

**Ex 1**: Let $f$ be the function from $\{a,b,c,d\}$ to $\{1,2,3\}$ defined by $f(a) = 3$, $f(b) = 2$, $f(c) = 1$, and $f(d) = 3$. Is $f$ a surjective (onto) function?

**Solution**: Yes, $f$ is surjective since all three elements of the codomain are images of elements in the domain. If the codomain were changed to $\{1,2,3,4\}$, $f$ would not be onto.

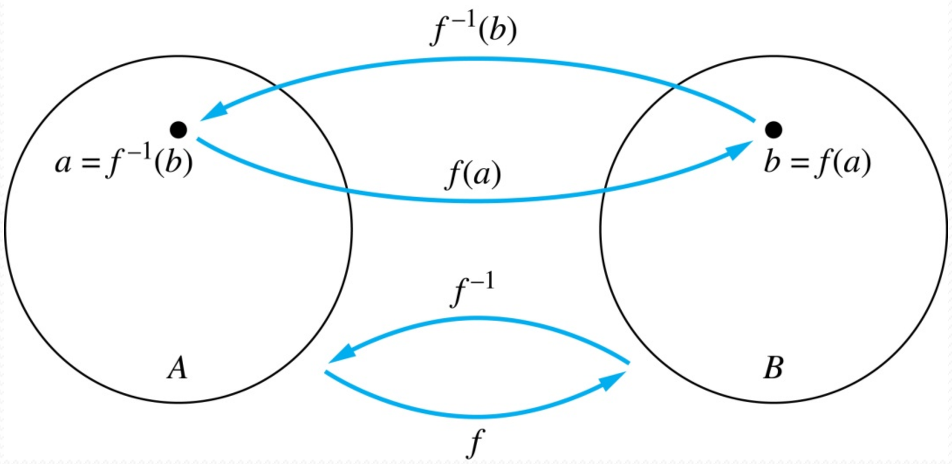
**Ex 2**: Is the function $f$: **Z** → **Z**, where $f(x) = x^2$ surjective?

**Solution**: No, $f$ is not surjective because there is no integer $x$ with $x^2 = -1$, for example.

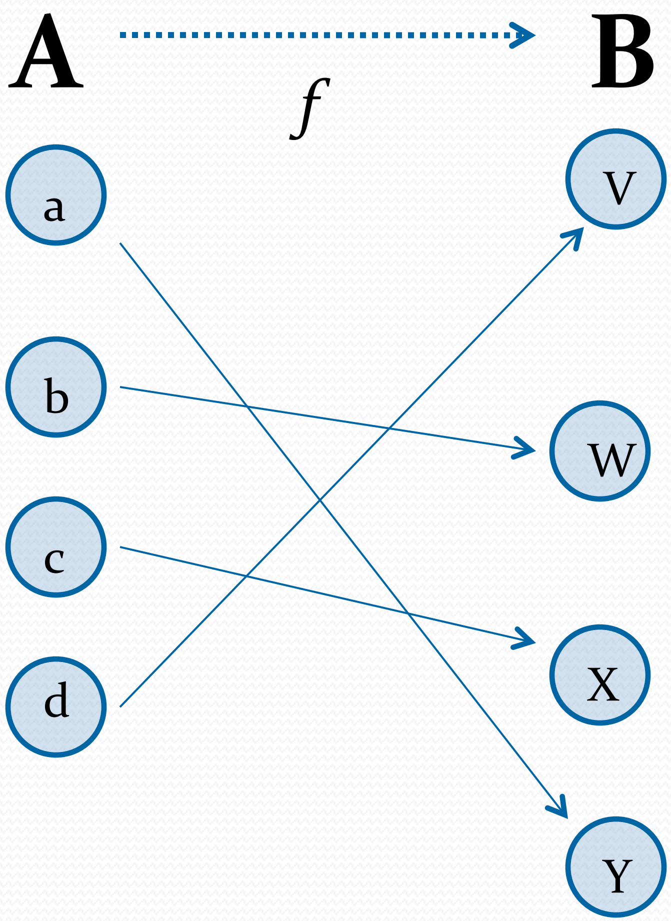
# Inverse Functions

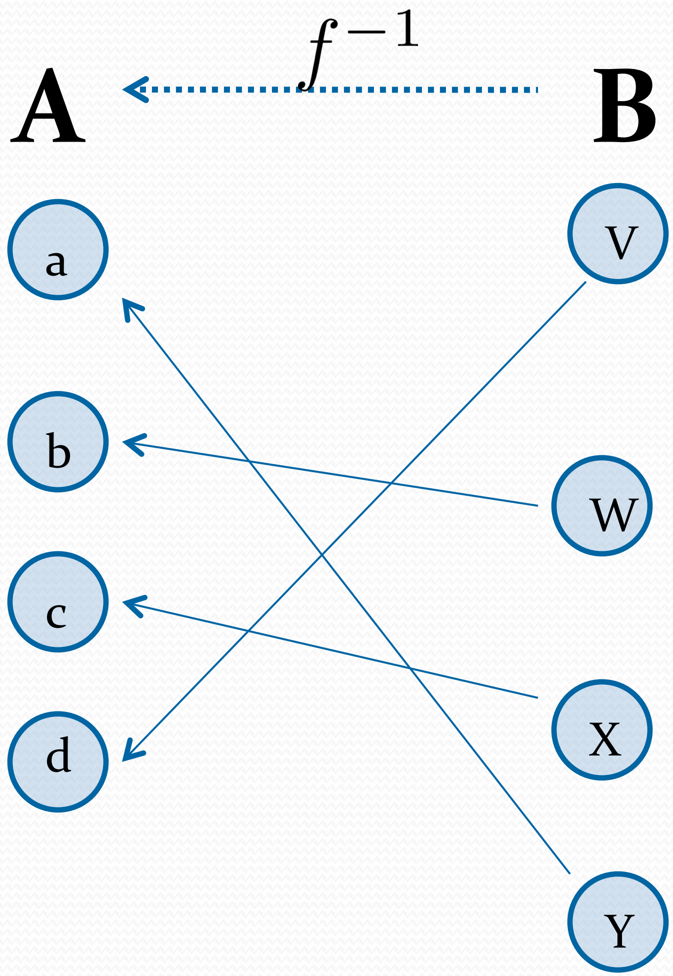
**Definition**: Let *f* be a bijection from *A* to *B*. Then the *inverse* of *f*, denoted $f^{-1}$, is the function from *B* to *A* defined as $f^{-1}(y) = x \text{ iff } f(x) = y$

No inverse exists unless *f* is a bijection. Why?



# Inverse Functions



# Questions

**Ex 1**: Let $f$ be the function from $\{a,b,c\}$ to $\{1,2,3\}$ such that $f(a) = 2$, $f(b) = 3$, and $f(c) = 1$. Is f invertible and if so what is its inverse?

**Solution**: The function $f$ is invertible because it both injective and surjective. The inverse function $f^{-1}$ reverses the correspondence given by $f$, so $f^{-1}(1) = c$, $f^{-1}(2) = a$, and $f^{-1}(3) = b$.

# Questions

**Ex 2**: Let $f$: **Z** → **Z** be such that $f(x) = x + 1$. Is $f$ invertible, and if so, what is its inverse?

**Solution**: The function $f$ is invertible because it is a bijection. The inverse function $f^{-1}$ reverses the correspondence so $f^{-1}(y) = y - 1$.

# Questions

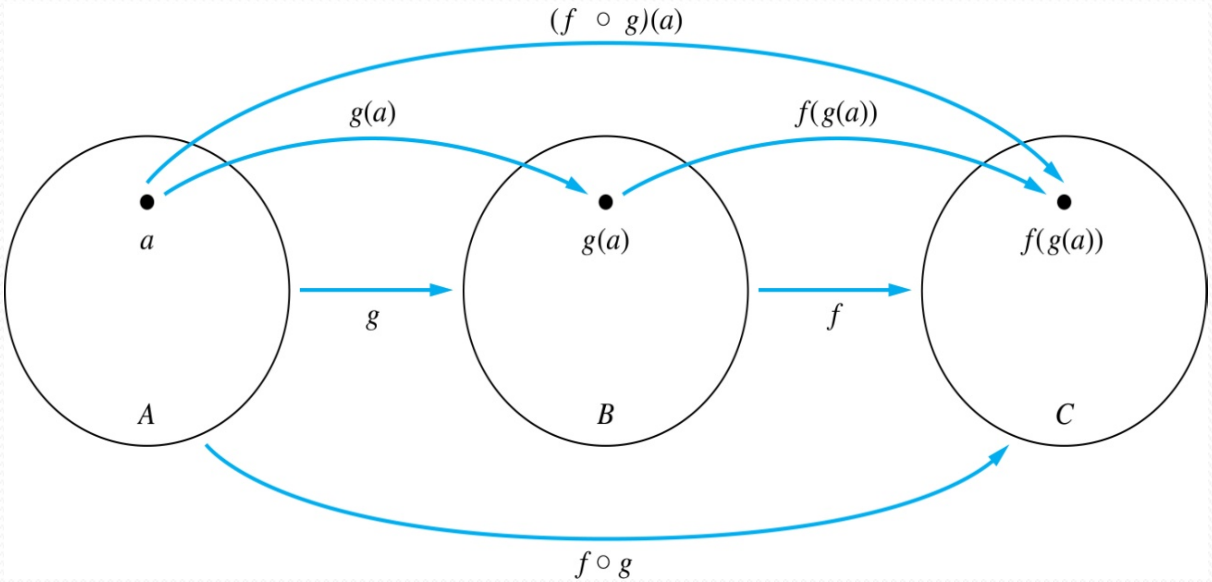
**Ex 3**: Let $f$: **R** → **R** be such that $f(x) = x^2$. Is $f$ invertible, and if so, what is its inverse?

**Solution**: The function $f$ is not invertible because it is not surjective.

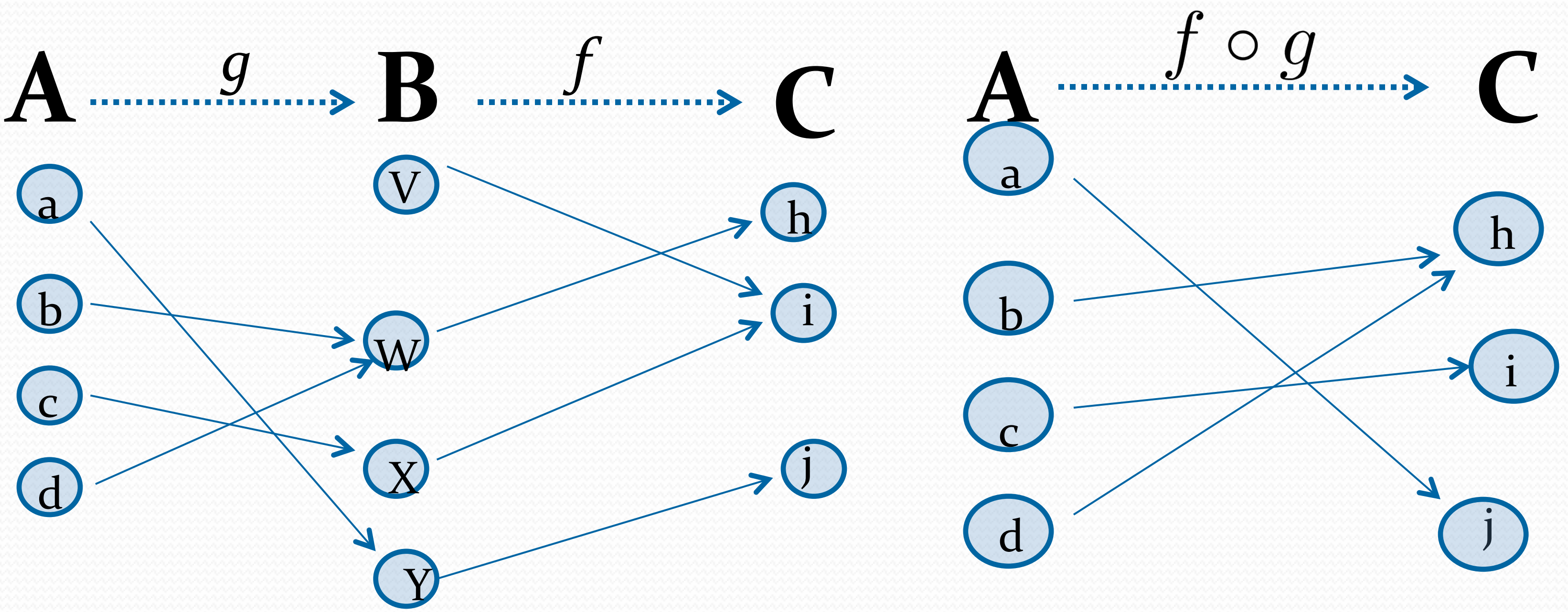
# Composition

- **Definition**: Let $f$: $B \rightarrow C$, $g$: $A \rightarrow B$. The *composition of f with g*, denoted $f \circ g$ is the function from $A$ to $C$ defined by

$$(f \circ g)(a) = f(g(a))$$



# Composition



# Composition Questions

**Ex 1**: If $f(x) = x^2$ and $g(x) = 2x + 1$, then

and $$f(g(x)) = (2x+1)^2$$

$$g(f(x)) = 2x^2 + 1$$

# Composition Questions

**Ex 2**: Let $g$ be the function from the set $\{a,b,c\}$ to itself such that $g(a) = b$, $g(b) = c$, and $g(c) = a$. Let $f$ be the function from the set $\{a,b,c\}$ to the set $\{1,2,3\}$ such that $f(a) = 3$, $f(b) = 2$, and $f(c) = 1$.

What is the composition of $f$ with $g$, and what is the composition of $g$ with $f$.

**Solution:** The composition $f \circ g$ is defined by

$f \circ g\ (a) = f(g(a)) = f(b) = 2.$

$f \circ g\ (b) = f(g(b)) = f(c) = 1.$

$f \circ g\ (c) = f(g(c)) = f(a) = 3.$

Note that $g \circ f$ is not defined, because the range of $f$ is not a subset of the domain of $g$.

# Composition Questions

**Ex 2**: Let f and g be functions from the set of integers to the set of integers defined by $f(x)= 2x + 3$ and $g(x)= 3x + 2$.

What is the composition of $f$ with $g$, and also the composition of $g$ with $f$ ?

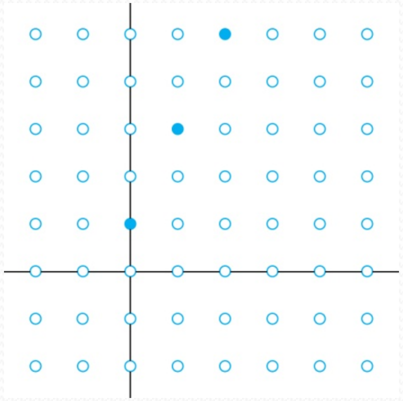
**Solution:**

$f \circ g\ (x) = f(g(x)) = f(3x + 2) = 2(3x + 2) + 3 = 6x + 7$

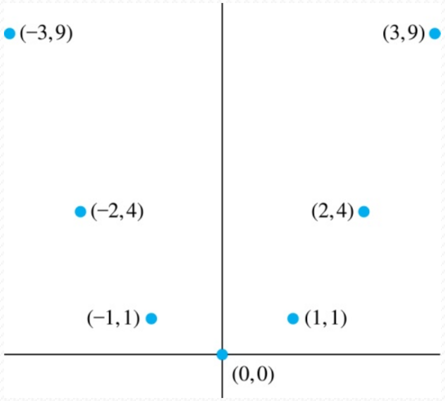
$g \circ f\ (x) = g(f(x)) = g(2x + 3) = 3(2x + 3) + 2 = 6x + 11$

# Graphs of Functions

- Let $f$ be a function from the set $A$ to the set $B$. The *graph* of the function $f$ is the set of ordered pairs $\{(a,b) \mid a \in A \text{ and } f(a) = b\}$.



Graph of $f(n) = 2n + 1$ from Z to Z




Graph of $f(x) = x^2$ from Z to Z

# Some Important Functions

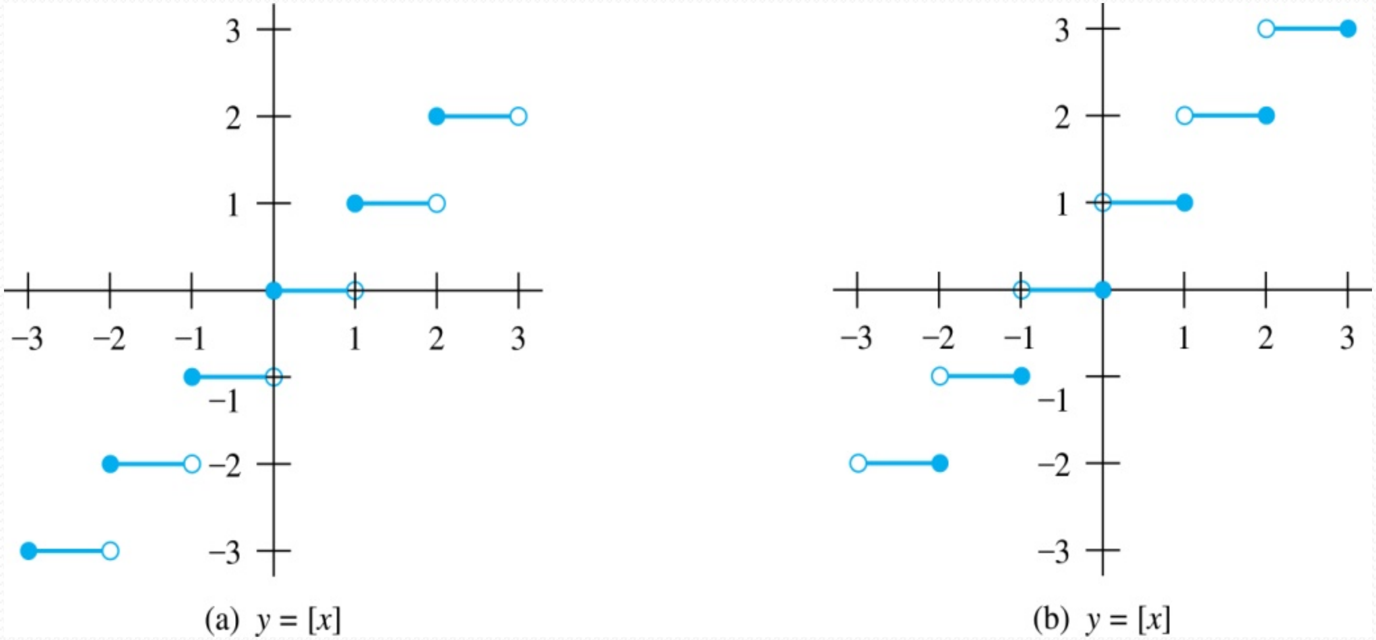
- The *floor* function, denoted $f(x) = \lfloor x \rfloor$
  is the largest integer less than or equal to *x*.

- The *ceiling* function, denoted $f(x) = \lceil x \rceil$
  is the smallest integer greater than or equal to *x*

**Ex:** $\lceil 3.5 \rceil = 4$ $\lfloor 3.5 \rfloor = 3$

$\lceil -1.5 \rceil = -1$ $\lfloor -1.5 \rfloor = -2$

# Floor and Ceiling Functions




Graph of (a) Floor and (b) Ceiling Functions

# Floor and Ceiling Functions

**TABLE 1 Useful Properties of the Floor and Ceiling Functions.**
($n$ is an integer, $x$ is a real number)

| | |
|---|---|
| (1a) | $\lfloor x \rfloor = n$ if and only if $n \le x < n + 1$ |
| (1b) | $\lceil x \rceil = n$ if and only if $n - 1 < x \le n$ |
| (1c) | $\lfloor x \rfloor = n$ if and only if $x - 1 < n \le x$ |
| (1d) | $\lceil x \rceil = n$ if and only if $x \le n < x + 1$ |
| (2) | $x - 1 < \lfloor x \rfloor \le x \le \lceil x \rceil < x + 1$ |
| (3a) | $\lfloor -x \rfloor = -\lceil x \rceil$ |
| (3b) | $\lceil -x \rceil = -\lfloor x \rfloor$ |
| (4a) | $\lfloor x + n \rfloor = \lfloor x \rfloor + n$ |
| (4b) | $\lceil x + n \rceil = \lceil x \rceil + n$ |

# Proving Properties of Functions

**Ex**: Prove that if x is a real number, then

$$\lfloor 2x \rfloor = \lfloor x \rfloor + \lfloor x + 1/2 \rfloor$$

**Solution**: Let $x = n + \varepsilon$, where $n$ is an integer and $0 \leq \varepsilon < 1$.

*Case 1*: $0 \leq \varepsilon < ½$

- $2x = 2n + 2\varepsilon$ and $\lfloor 2x \rfloor = 2n$, since $0 \leq 2\varepsilon < 1$.
- $\lfloor x + 1/2 \rfloor = n$, since $x + ½ = n + (1/2 + \varepsilon)$ and $0 \leq ½ + \varepsilon < 1$.
- Hence, $\lfloor 2x \rfloor = 2n$ and $\lfloor x \rfloor + \lfloor x + 1/2 \rfloor = n + n = 2n$.

*Case 2*: $½ \leq \varepsilon < 1$

- $2x = 2n + 2\varepsilon = (2n + 1) + (2\varepsilon - 1)$ and $\lfloor 2x \rfloor = 2n + 1$, since $0 \leq 2\varepsilon - 1 < 1$.
- $\lfloor x + 1/2 \rfloor = \lfloor n + (1/2 + \varepsilon) \rfloor = \lfloor n + 1 + (\varepsilon - 1/2) \rfloor = n + 1$ since $0 \leq \varepsilon - 1/2 < 1$.
- Hence, $\lfloor 2x \rfloor = 2n + 1$ and $\lfloor x \rfloor + \lfloor x + 1/2 \rfloor = n + (n + 1) = 2n + 1$. ◀

# Factorial Function

**Definition:** The factorial function $f$: $\mathbf{N} \to \mathbf{Z}^+$, denoted by $f(n) = n!$ is the product of the first $n$ positive integers when $n$ is a nonnegative integer.

$$f(n) = 1 \cdot 2 \cdots (n-1) \cdot n, \qquad f(0) = 0! = 1$$

**Examples:**

$$f(1) = 1! = 1$$

$$f(2) = 2! = 1 \cdot 2 = 2$$

Stirling's Formula:

$$n! \sim \sqrt{2\pi n}(n/e)^n$$

$$f(n) \sim g(n) \doteq lim_{n\to\infty} f(n)/g(n) = 1$$

$$f(6) = 6! = 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 = 720$$

$$f(20) = 2{,}432{,}902{,}008{,}176{,}640{,}000.$$