Homework 7

Keep in mind that n is a positive integer if $n \geq 1$.

Section 5.1

4. Let $P(n)$ be the statement $1^3 + 2^3 + ... + n^3 = (\frac{n(n+1)}{2})^2$ for any positive integer n. Prove this using mathematical induction (which is done in parts below).
   (a) What is the basis step? Prove this.
   (b) What is the inductive hypothesis?
   (c) What is the inductive step? Prove this, and state your assumptions.
   (d) Explain why these steps show that this formula is true for whenever n is a positive integer.

10. (a) Find a formula for $\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + ... + \frac{1}{n(n+1)}$ by examining the values of this expression for small values of n (use forward or backward substitution to help).
    (b) Prove the formula you conjectured in part (a) is correct using mathematical induction.

18. Let $P(n)$ be the statement that $n! < n^n$. Prove using mathematical induction that $P(n)$ is true for any integer $n \geq 2$.

32. Prove using mathematical induction that 3 divides $n^3 + 2n$ whenever n is a positive integer.

38. Prove that if $A_1, A_2, ..., A_n$ and $B_1, B_2, ..., B_n$ are sets such that $A_j \subseteq B_j$ for $j = 1, 2, ..., n$, then $\bigcup_{j=1}^{n} A_j \subseteq \bigcup_{j=1}^{n} B_j$.

Section 5.3

4. Find $f(2)$, $f(3)$, $f(4)$, and $f(5)$ if f is defined recursively by $f(0) = f(1) = 1$ and for $n = 1, 2, ...$
   (a) $f(n+1) = f(n) - f(n-1)$
   (b) $f(n+1) = f(n) \cdot f(n-1)$
   (c) $f(n+1) = f(n)^2 + f(n-1)^3$
   (d) $f(n+1) = \frac{f(n)}{f(n-1)}$

12. Let f_n be the nth Fibonacci number. Prove using mathematical induction that $f_1^2 + f_2^2 + ... + f_n^2 = f_n \cdot f_{n+1}$ when n is a positive integer.

24. Give a recursive definition of
   (a) the set of odd positive integers (i.e., $\{1, 3, 5, 7, ...\}$).
   (b) the set of positive integer powers of 3 (i.e., $\{3, 9, 27, 81, ...\}$)

44. The set of leaves and the set of internal vertices of a full binary tree can be defined recursively.

    Basis step: The root r is a leaf of the full binary tree with exactly one vertex r. This tree has no internal vertices.

    Recursive step: The set of leaves of the tree $T = T_1 \cdot T_2$ is the union of the sets of leaves of T_1 and of T_2. The internal vertices of T are the root r of T and the union of the set of internal vertices of T_1 and the set of internal vertices of T_2.

    Use structural induction to show that $l(t)$, which is the number of leaves of a full binary tree T, is 1 more than $i(T)$, which is the number of internal vertices of T.