# Selection

# Selection Problem

- Given an integer *k* and *n* elements $x_1, x_2, \ldots, x_n$, taken from a total order, find the *k*-th smallest element in this set.

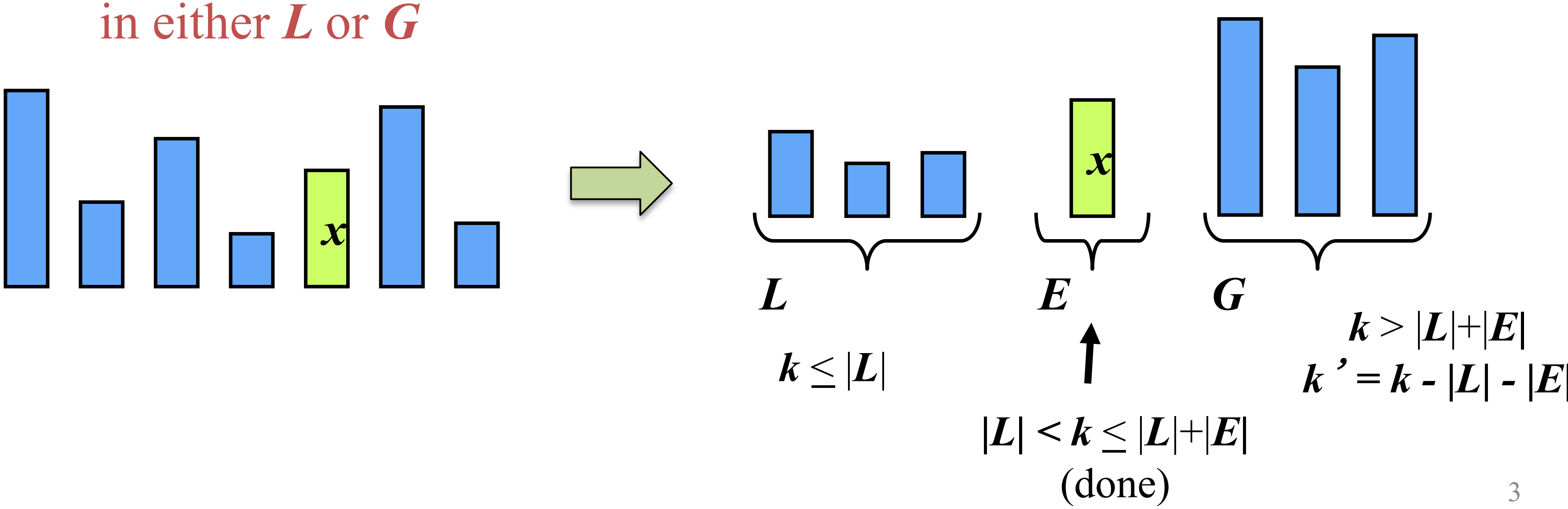
- Of course, we can sort the set in O(*n* log *n*) time and then index the *k*-th element.
  - Ex when k=3:
    5, 10, 6, 3, 14, 12, 2 → 2, 3, 5, 6, 10, 12, 14

- Can we solve the selection problem faster?

# Quick-Select

A randomized selection algorithm based on the prune-and-search paradigm:

- Prune: pick a random element $\boldsymbol{x}$ (called pivot) and partition $\boldsymbol{S}$ into
  - $\boldsymbol{L}$ elements less than $\boldsymbol{x}$
  - $\boldsymbol{E}$ elements equal $\boldsymbol{x}$
  - $\boldsymbol{G}$ elements greater than $\boldsymbol{x}$
- Search: depending on $k$, either answer is in $\boldsymbol{E}$, or we need to recurse in either $\boldsymbol{L}$ or $\boldsymbol{G}$



# Partition

We partition an input sequence as in the quick-sort algorithm:

- Remove, in turn, each element $y$ from $S$ and
- Insert $y$ into $L$, $E$ or $G$, depending on the result of the comparison with the pivot $p$

Each insertion and removal takes $O(1)$ time

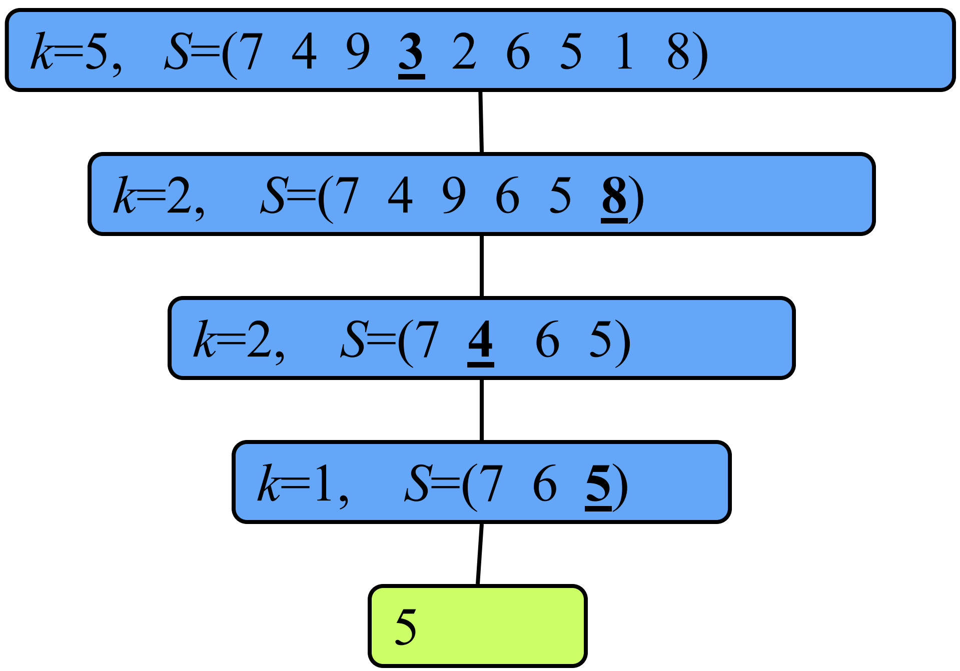
Thus, the partition step of quick-select takes $O(n)$ time

**Algorithm** *partition*($S$, $p$)
    **Input** sequence $S$, pivot $p$
    **Output** subsequences $L$, $E$, $G$ of the elements of $S$ less than, equal to, or greater than the pivot, resp.
    $L, E, G \leftarrow$ empty sequences
    **while** $\neg S.isEmpty()$
        $y \leftarrow S.remove(S.first())$
        **if** $y < p$
            $L.insertLast(y)$
        **else if** $y = p$
            $E.insertLast(y)$
        **else** $\{ y > p \}$
            $G.insertLast(y)$
    **return** $L, E, G$

# Quick-Select Visualization

An execution of quick-select can be visualized by a recursion path

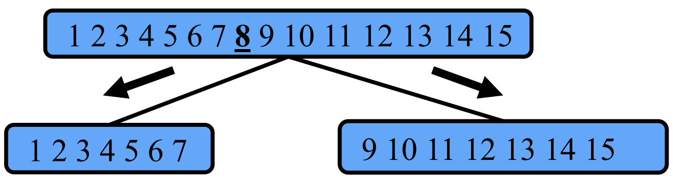
- each node represents a recursive call of quick-select, and stores $k$ and the remaining sequence



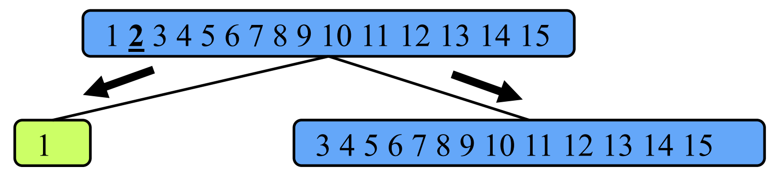
# Expected Running Time

Consider a recursive call of quick-select on a sequence of size $s$

- **Good call:** the sizes of $L$ and $G$ are each less than $3s/4$
- **Bad call:** one of $L$ and $G$ has size greater than $3s/4$



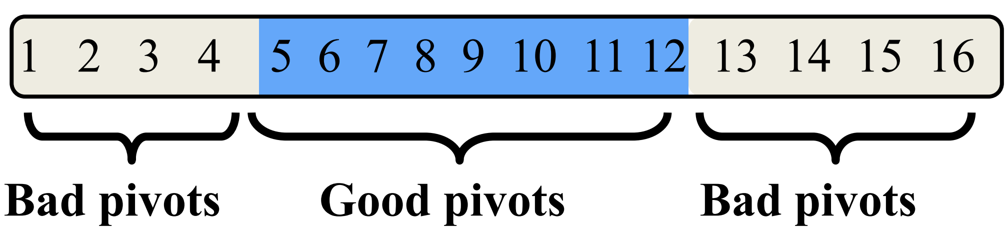

**Good call**




**Bad call**

A call is good with probability 1/2

- 1/2 of the possible pivots cause good calls:



# Expected Running Time (2)

Probabilistic Fact #1: The expected number of coin tosses required in order to get one head is two.

Probabilistic Fact #2: Expectation is a linear function:

- $E(X + Y) = E(X) + E(Y)$
- $E(cX) = cE(X)$

Let $T(n)$ denote the <u>expected</u> running time of quick-select.

- By Fact #2,
  - $T(n) \leq T(3n/4) + bn$*(expected # of calls before a good call)
- By Fact #1,
  - $T(n) \leq T(3n/4) + 2bn$
- That is, $T(n)$ is a geometric series:
  - $T(n) \leq 2bn + 2b(3/4)n + 2b(3/4)^2n + 2b(3/4)^3n + \ldots$
- So $T(n)$ is $O(n)$.

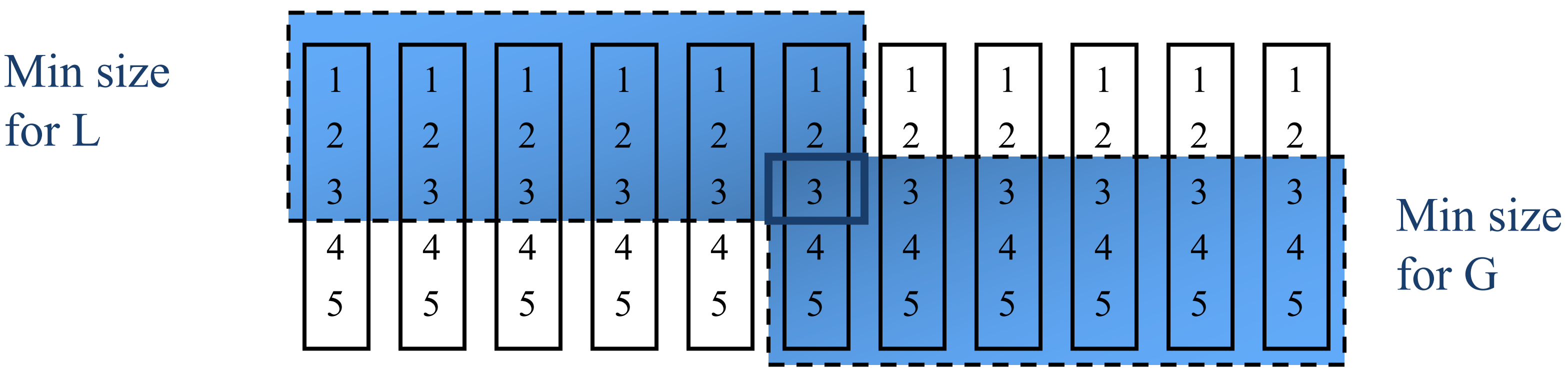
Randomized quick-select runs in $O(n)$ expected time.

# Deterministic Selection

We *can* do selection in $O(n)$ worst-case time.

Main idea: recursively use the selection algorithm itself to find a good pivot for quick-select

- Divide S into $n/5$ sets of 5 each
- Find a median in each set
- Recursively find the median of the "baby" medians.
- Use median of medians as a guaranteed good pivot




See Exercise C-4.24 for details of analysis.