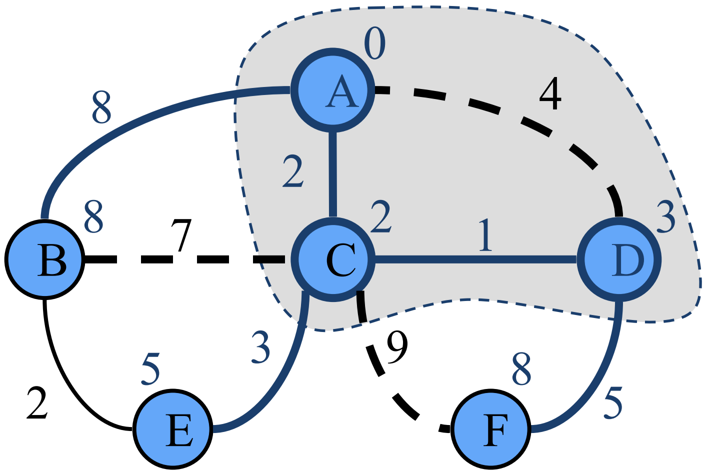
Shortest Paths

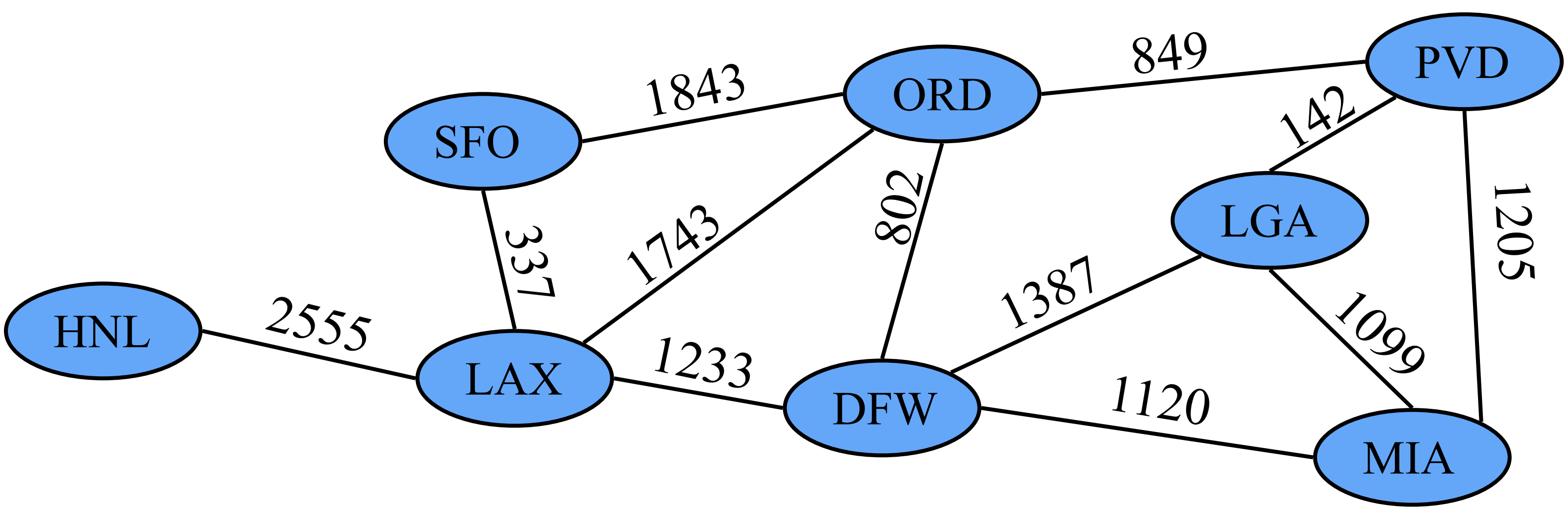


Outline and Reading

- Weighted graphs (7.1)
  - Shortest path problem
  - Shortest path properties
- Dijkstra's algorithm (7.1.1)
  - Algorithm
  - Edge relaxation
- The Bellman-Ford algorithm (7.1.2)
- Shortest paths in DAGs (7.1.3)
- All-pairs shortest paths (7.2.1)

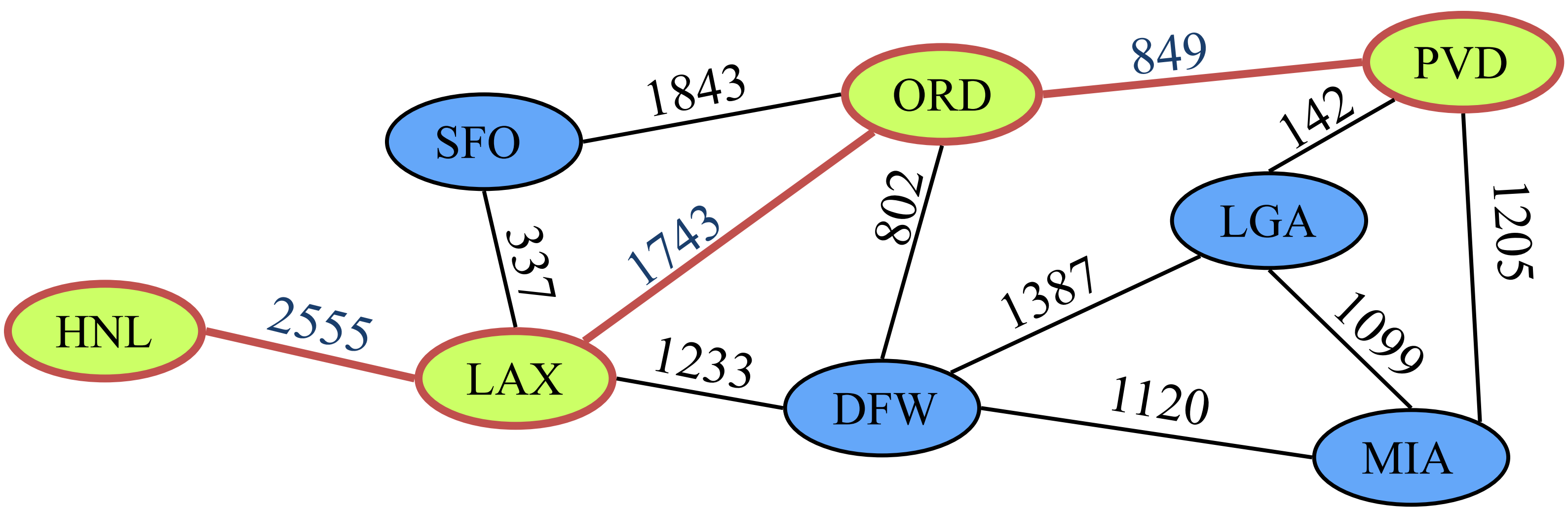
Weighted Graphs

- In a weighted graph, each edge has an associated numerical value, called the weight of the edge
- Edge weights may represent, distances, costs, etc.
- Example:
  - In a flight route graph, the weight of an edge represents the distance in miles between the endpoint airports



Shortest Path Problem

- Given a weighted graph and two vertices ***u*** and ***v***, we want to find a path of minimum total weight between ***u*** and ***v.***
  - Length of a path is the sum of the weights of its edges
- Example: shortest path between Providence and Honolulu
- Applications
  - Internet packet routing
  - Flight reservations
  - Driving directions

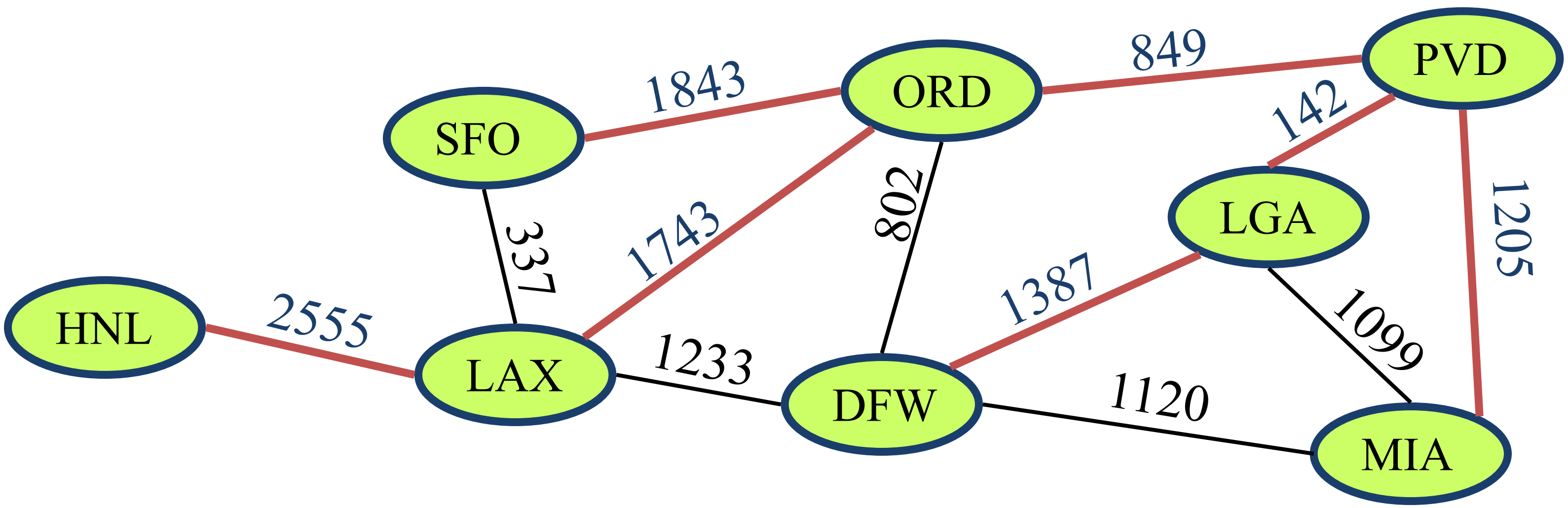


Shortest Path Problem

Property 1. A subpath of a shortest path is itself a shortest path.

Property 2. There is a tree of shortest paths from a start vertex to all other vertices.

- Example: tree of shortest paths from Providence



Dijkstra's Algorithm

The distance of vertex v from s is the length of a shortest path between s and v.

Dijkstra's algorithm computes the distances of all the vertices from a given start vertex s.

- Assumptions:
  - the graph is connected
  - the edges are undirected
  - the edge weights are nonnegative

Idea:

- Grow a "cloud" of vertices, beginning with s and eventually covering all vertices
- Store with each vertex v a label $d(v)$ representing the distance of v from s in the subgraph consisting of the cloud and its adjacent vertices
- At each step
  - Add to the cloud the vertex u outside the cloud with the smallest distance label, $d(u)$
  - Update the labels of the vertices adjacent to u

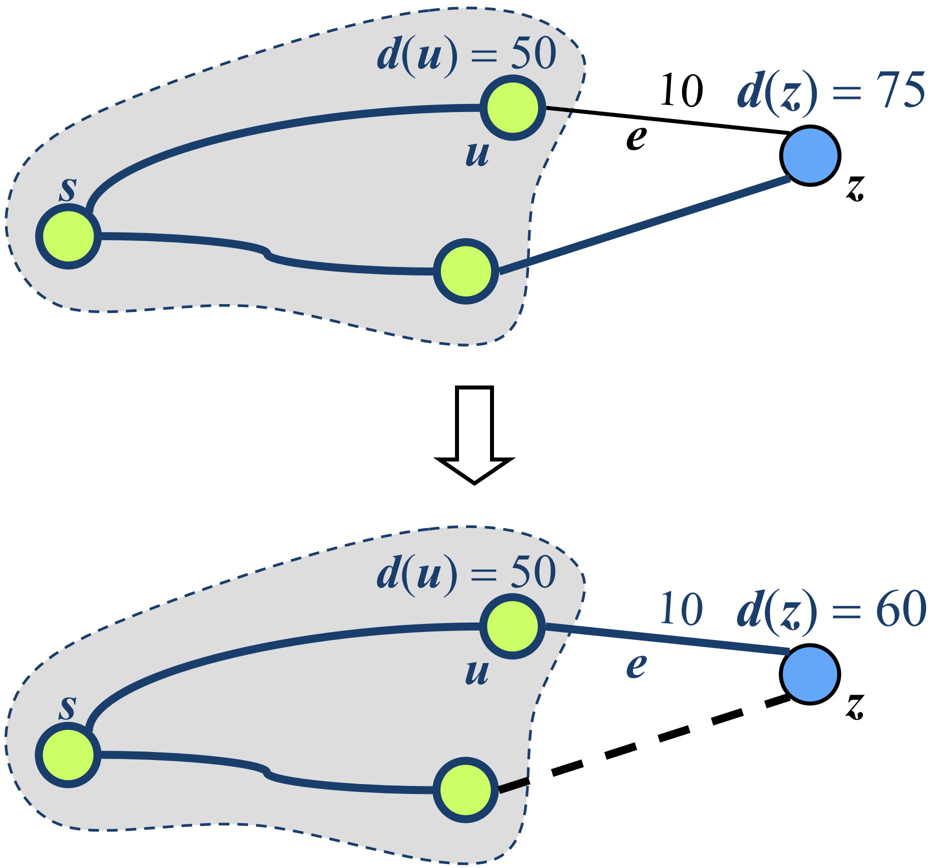
Edge Relaxation

Consider an edge $e = (u,z)$ such that

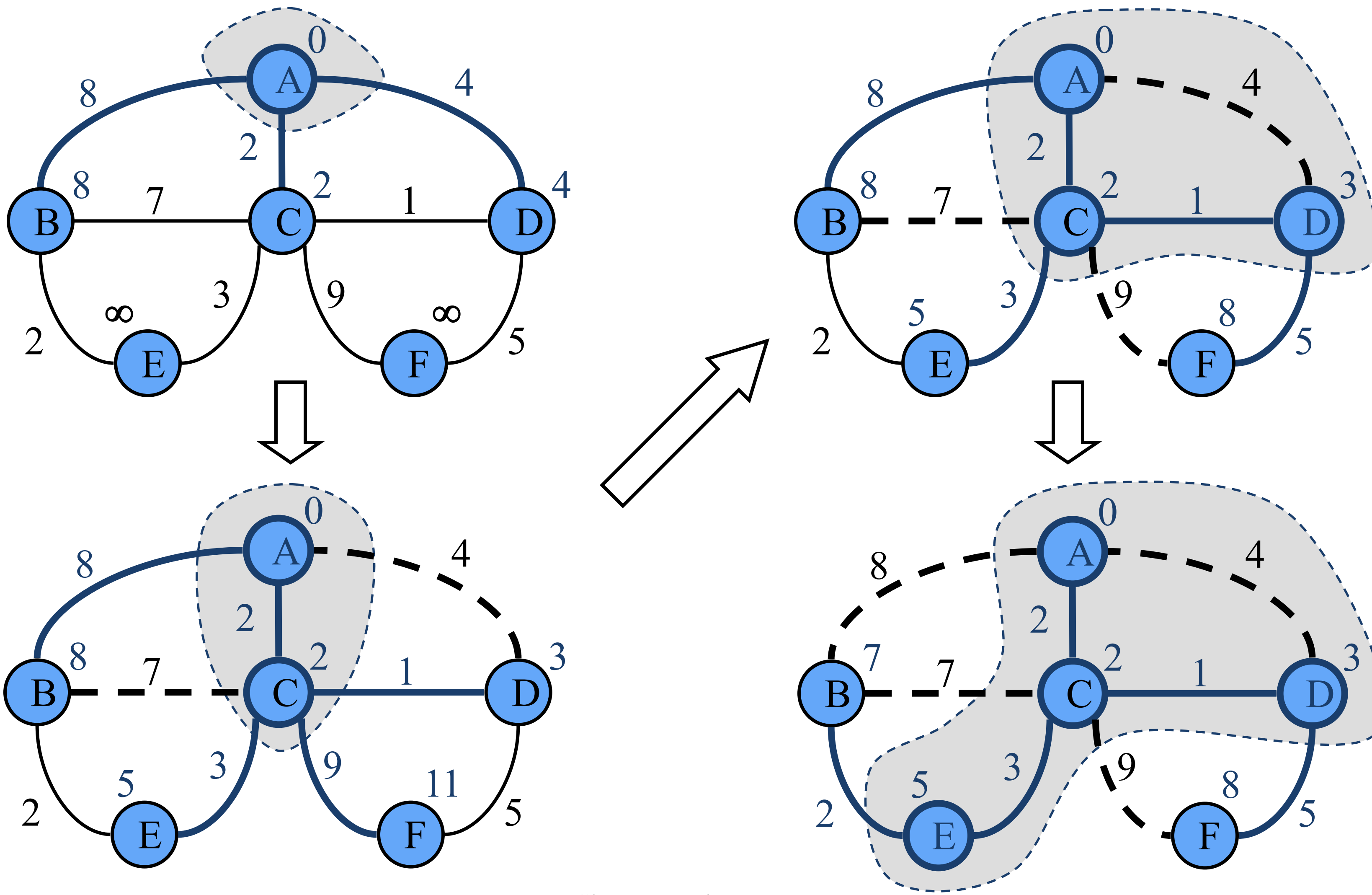
- u is the vertex most recently added to the cloud
- z is not in the cloud

The relaxation of edge e updates distance $d(z)$ as follows:

$$d(z) \leftarrow \min\{d(z), d(u) + weight(e)\}$$



Example



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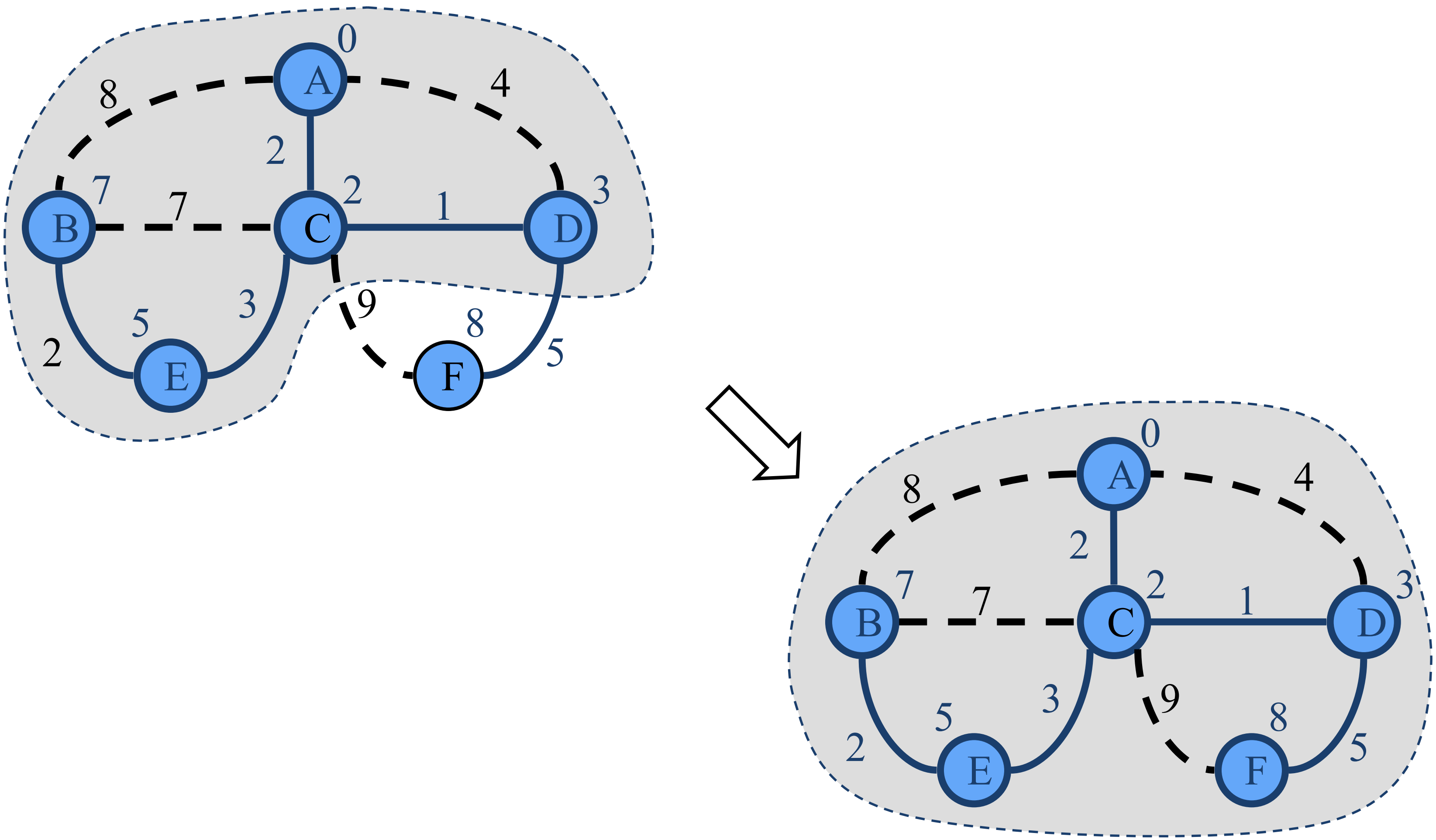
Example (cont.)



Dijkstra's Algorithm

A priority queue stores the vertices outside the cloud

- Key: distance
- Element: vertex

Locator-based methods

- *insert*(*k*,*e*) returns a locator
- *replaceKey*(*l*,*k*) changes the key of an item

We store two labels with each vertex:

- Distance (*d(v)* label)
- locator in priority queue

```
Algorithm DijkstraDistances(G, s)
Q ← new heap-based priority queue
for all v ∈ G.vertices()
    if v = s
        setDistance(v, 0)
    else
        setDistance(v, ∞)
    l ← Q.insert(getDistance(v), v)
    setLocator(v,l)
while ¬Q.isEmpty()
    u ← Q.removeMin()
    for all e ∈ G.incidentEdges(u)
        { relax edge e }
        z ← G.opposite(u,e)
        r ← getDistance(u) + weight(e)
        if r < getDistance(z)
            setDistance(z,r)
            Q.replaceKey(getLocator(z),r)
```

Analysis

- Graph operations
  - Method incidentEdges is called once for each vertex
- Label operations
  - We set/get the distance and locator labels of vertex z $O(\deg(z))$ times
  - Setting/getting a label takes $O(1)$ time
- Priority queue operations
  - Each vertex is inserted once into and removed once from the priority queue, where each insertion or removal takes $O(\log n)$ time
  - The key of a vertex in the priority queue is modified at most $\deg(w)$ times, where each key change takes $O(\log n)$ time
- Dijkstra's algorithm runs in $O((n + m) \log n)$ time provided the graph is represented by the adjacency list structure
  - Recall that $\Sigma_v \deg(v) = 2m$
- The running time can also be expressed as $O(m \log n)$ since the graph is connected.

Extension

Using the template method pattern, we can extend Dijkstra's algorithm to return a tree of shortest paths from the start vertex to all other vertices

- Store with each vertex a third label:
  - parent edge in the shortest path tree
- In the edge relaxation step, update the parent label

```
Algorithm DijkstraShortestPathsTree(G, s)

    …

  for all v ∈ G.vertices()
    …
    setParent(v, ∅)
    …

    for all e ∈ G.incidentEdges(u)
      { relax edge e }
      z ← G.opposite(u,e)
      r ← getDistance(u) + weight(e)
      if r < getDistance(z)
        setDistance(z,r)
        setParent(z,e)
        Q.replaceKey(getLocator(z),r)
```

Why Dijkstra's Algorithm Works

Dijkstra's algorithm is based on the greedy method. It adds vertices by increasing distance.

<u>Claim</u>: Whenever a vertex u is pulled into the cloud, $D[u] = d(v, u)$.

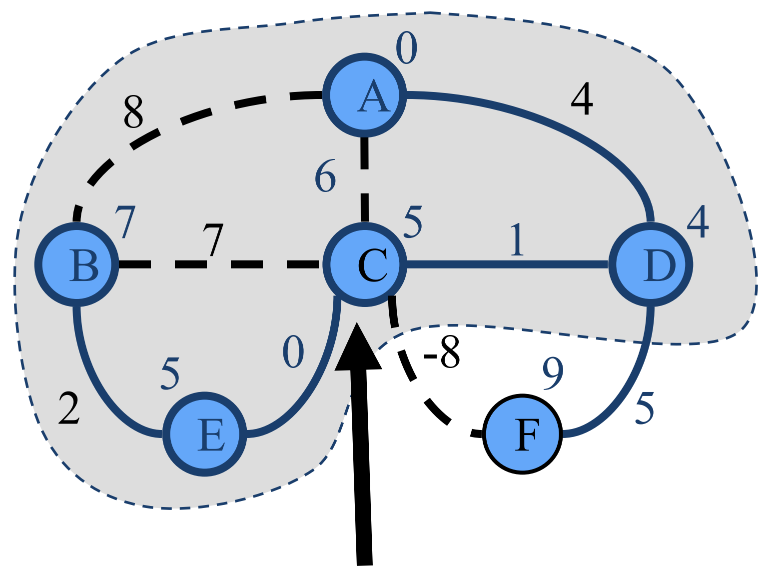
Outline of Proof (by contradiction):

- Suppose ***u*** is the **first** vertex such that $D[u] > d(v, u)$.
- Let ***z*** be the first vertex on the shortest v-u path P which hasn't been pulled into the cloud yet, and let ***y*** be the vertex before z on P.
  - Then, $D[z] = d(v, z)$.
  - Since z is on shortest v-u path, $d(v, z) + d(z, u) = d(v, u)$.
  - Since u is processed before z, $D[u] \leq D[z]$.
- $D[u] \leq D[z] = d(v, z) \leq d(v, z) + d(z, u) = d(v, u)$, a contradiction.

Why It Doesn't Work for Negative-Weight Edges

Dijkstra's algorithm is based on the greedy method. It adds vertices by increasing distance.

- If a node with a negative incident edge were to be added late to the cloud, it could mess up distances for vertices already in the cloud.
- This violates the greedy property.



C's true distance is 1, but it is already in the cloud with d(*C*)=5!

Bellman-Ford Algorithm

- Works even with negative-weight edges
- Must assume directed edges (otherwise we would have negative-weight cycles)
- Iteration ***i*** finds all shortest paths that use *i* edges beginning at ***s***

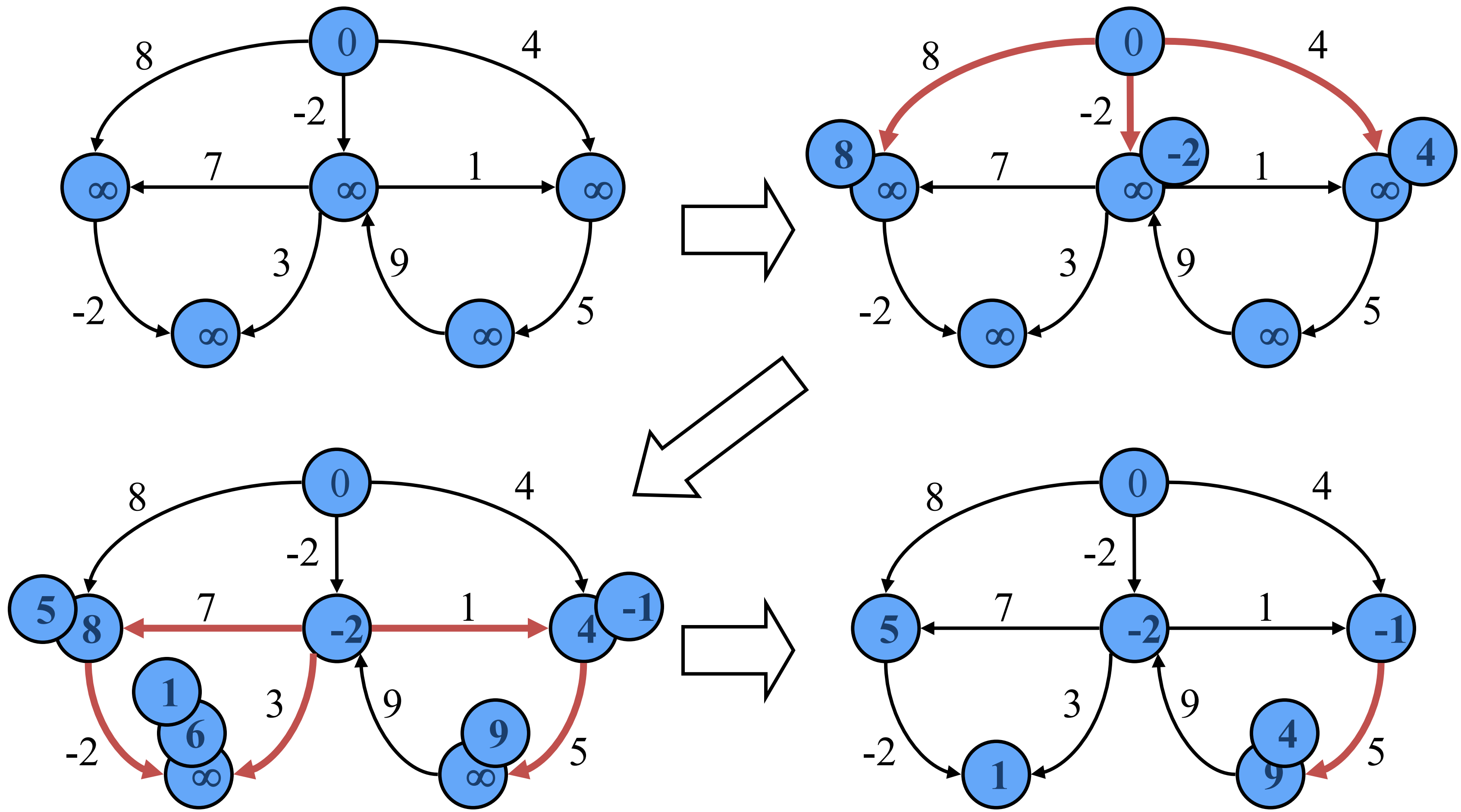
- Running time: $O(nm)$.

- Can be extended to detect a negative-weight cycle if it exists
  - How?

```
Algorithm BellmanFord(G, s)
  for all v ∈ G.vertices()
    if v = s
      setDistance(v, 0)
    else
      setDistance(v, ∞)
  for i ← 1 to n-1 do
    for each (directed) edge e=(u,z) ∈ G.edges()
      { relax edge e }
      r ← getDistance(u) + weight(e)
      if r < getDistance(z)
        setDistance(z,r)
```

Bellman-Ford Example

Nodes are labeled with their d(*v*) values



DAG-based Algorithm

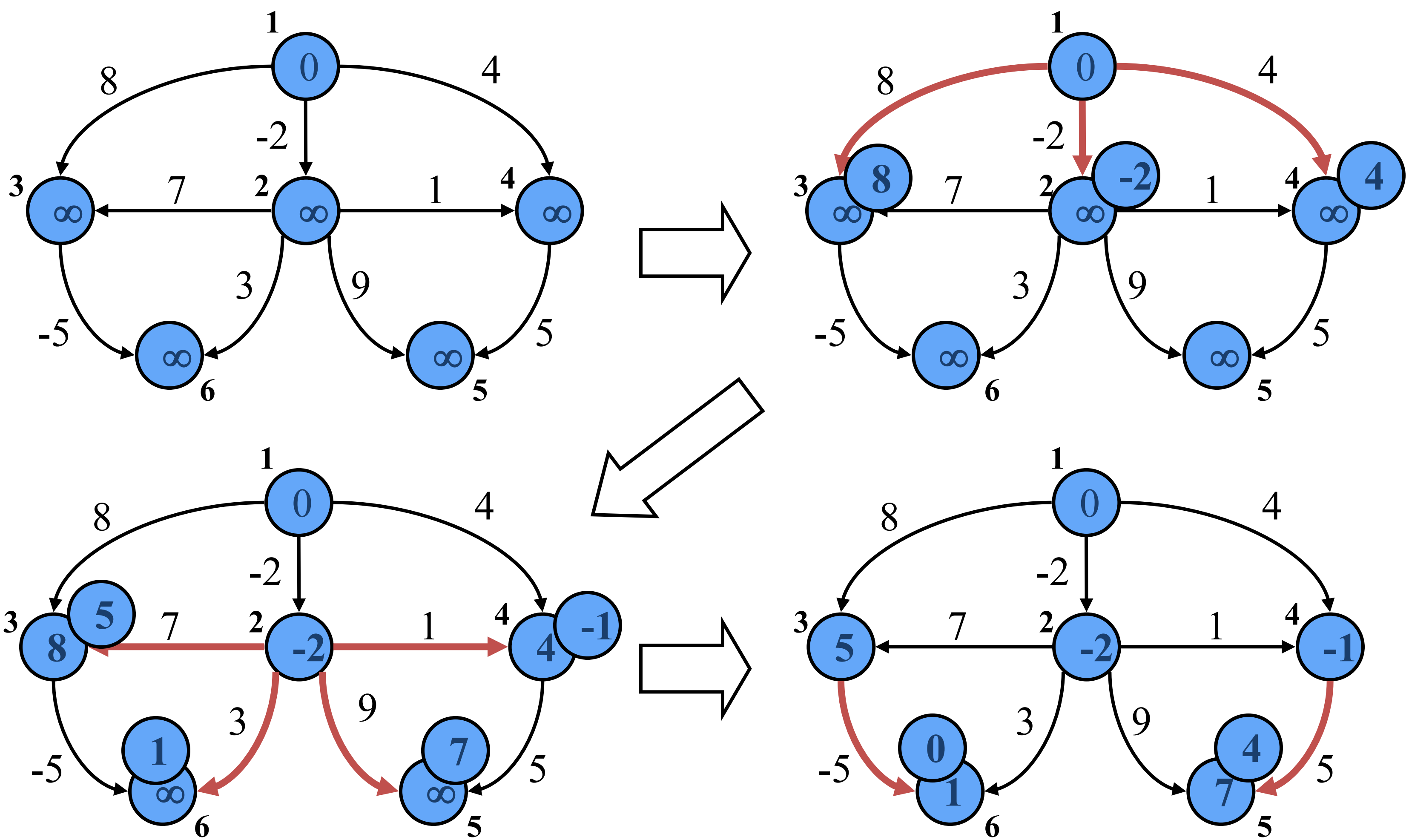
- Assumes *G* is a DAG
- Works even with negative-weight edges
- Uses topological order
- Much faster than Dijkstra's algorithm

- Running time: O(*n*+*m*).

```
Algorithm DagDistances(G, s)
  for all v ∈ G.vertices()
    if v = s
      setDistance(v, 0)
    else
      setDistance(v, ∞)
  Perform a topological sort of the vertices
  for u ← 1 to n do    {in topological order}
    for each edge e=(u,z) ∈ G.edges()
      { relax edge e }
      r ← getDistance(u) + weight(e)
      if r < getDistance(z)
        setDistance(z,r)
```

DAG Example

Nodes are labeled with their d(*v*) values



(two steps)

All-Pairs Shortest Paths

Find the distance between every pair of vertices in a weighted directed graph G.

- We can make n calls to Dijkstra's algorithm (if no negative edges), which takes O($nm\log n$) time.

- Likewise, n calls to Bellman-Ford would take O(n^2m) time.

We can achieve O(n^3) time using the Floyd-Warshall dynamic programming algorithm.

```
Algorithm AllPair(G) {assumes vertices 1,…,n}
 for all vertex pairs (i,j)
   if i = j
      D0[i,i] ← 0
   else if (i,j) is an edge in G
      D0[i,j] ← weight of edge (i,j)
   else
      D0[i,j] ← + ∞
 for k ← 1 to n do
   for i ← 1 to n do
     for j ← 1 to n do
       Dk[i,j] ← min{ Dk-1[i,j],  Dk-1[i,k]+Dk-1[k,j] }
 return Dn
```