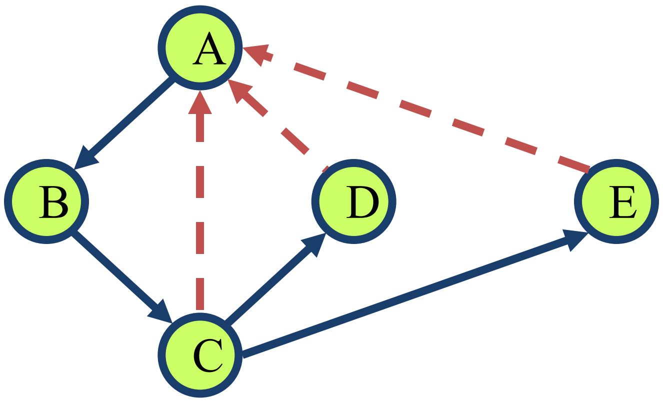
Depth-First Search



Outline and Reading

Definitions (6.1)

- Subgraph
- Connectivity
- Spanning trees and forests

Depth-first search (6.3.1)

- Algorithm
- Example
- Properties
- Analysis

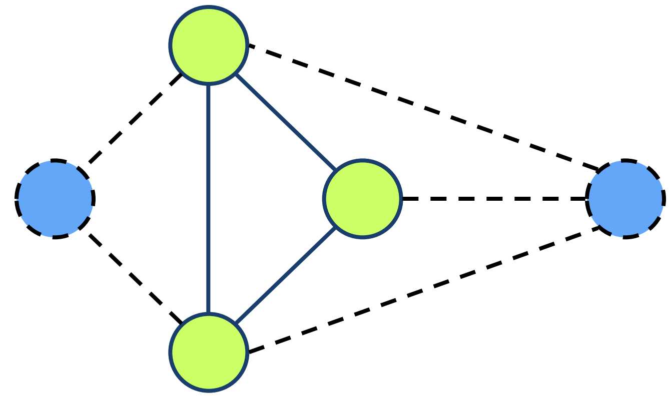
Applications of DFS (6.5)

- Path finding
- Cycle finding

Subgraphs

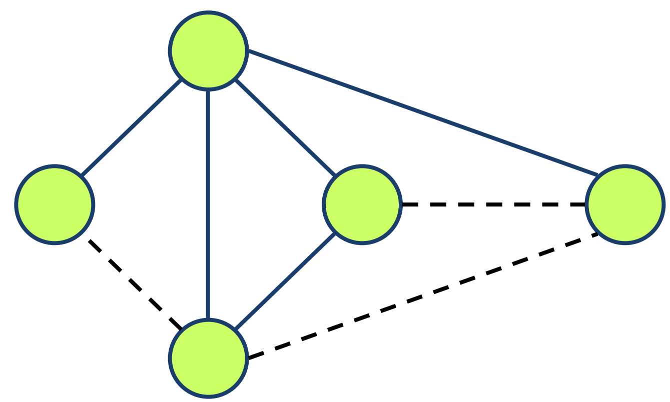
A subgraph S of a graph G is a graph such that

- the vertices of S are a subset of the vertices of G
- the edges of S are a subset of the edges of G



Subgraph

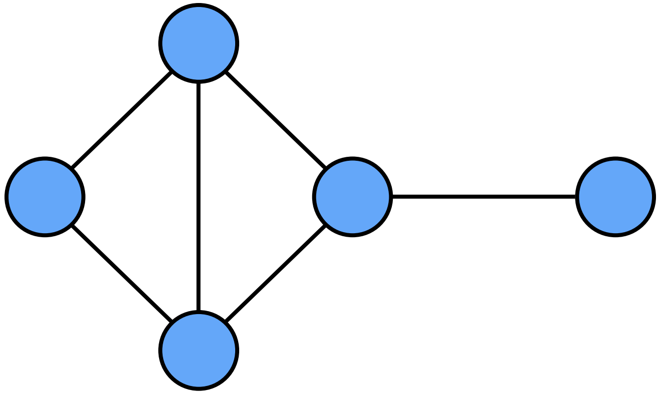
A spanning subgraph of G is a subgraph that contains all the vertices of G



Spanning subgraph

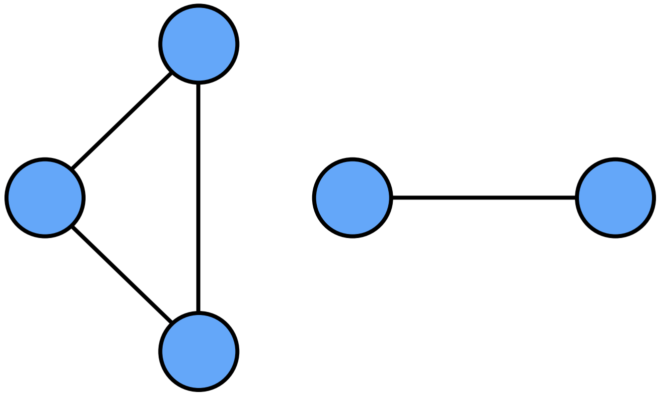
Connectivity

A graph is connected if there is a path between every pair of vertices



Connected graph

A connected component of a graph G is a maximal connected subgraph of G

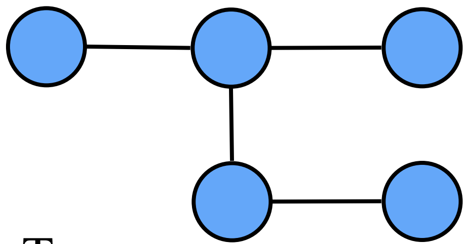


Non connected graph with two connected components

Trees and Forests

- A (free) tree is an undirected graph T such that
  - T is connected
  - T has no cycles

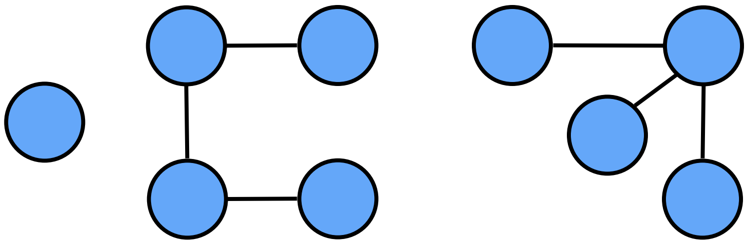
  This definition of tree is different from the one of a rooted tree



Tree

- A forest is an undirected graph without cycles

- The connected components of a forest are trees

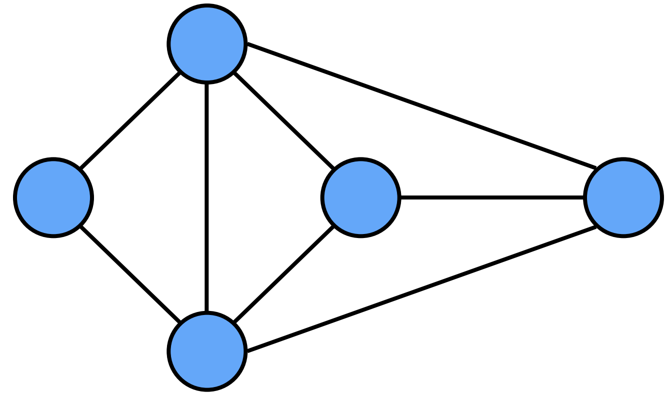


Forest

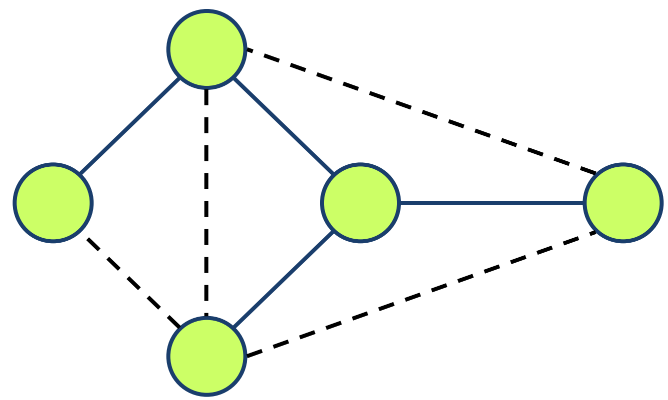
Spanning Trees and Forests

A spanning tree of a connected graph is a spanning subgraph that is a tree

- A spanning tree is not unique unless the graph is a tree
- Spanning trees have applications to the design of communication networks
- A spanning forest of a graph is a spanning subgraph that is a forest



Graph



Spanning tree

Depth-First Search

- Depth-first search (DFS) is a general technique for traversing a graph. A DFS traversal of a graph G
  - visits all the vertices and edges of G
  - determines whether G is connected
  - computes the connected components of G
  - computes a spanning forest of G
- DFS on a graph with n vertices and m edges takes $O(n + m)$ time
- DFS can be further extended to solve other graph problems
  - find and report a path between two given vertices
  - find a cycle in the graph
- Depth-first search is to graphs what Euler tour is to binary trees

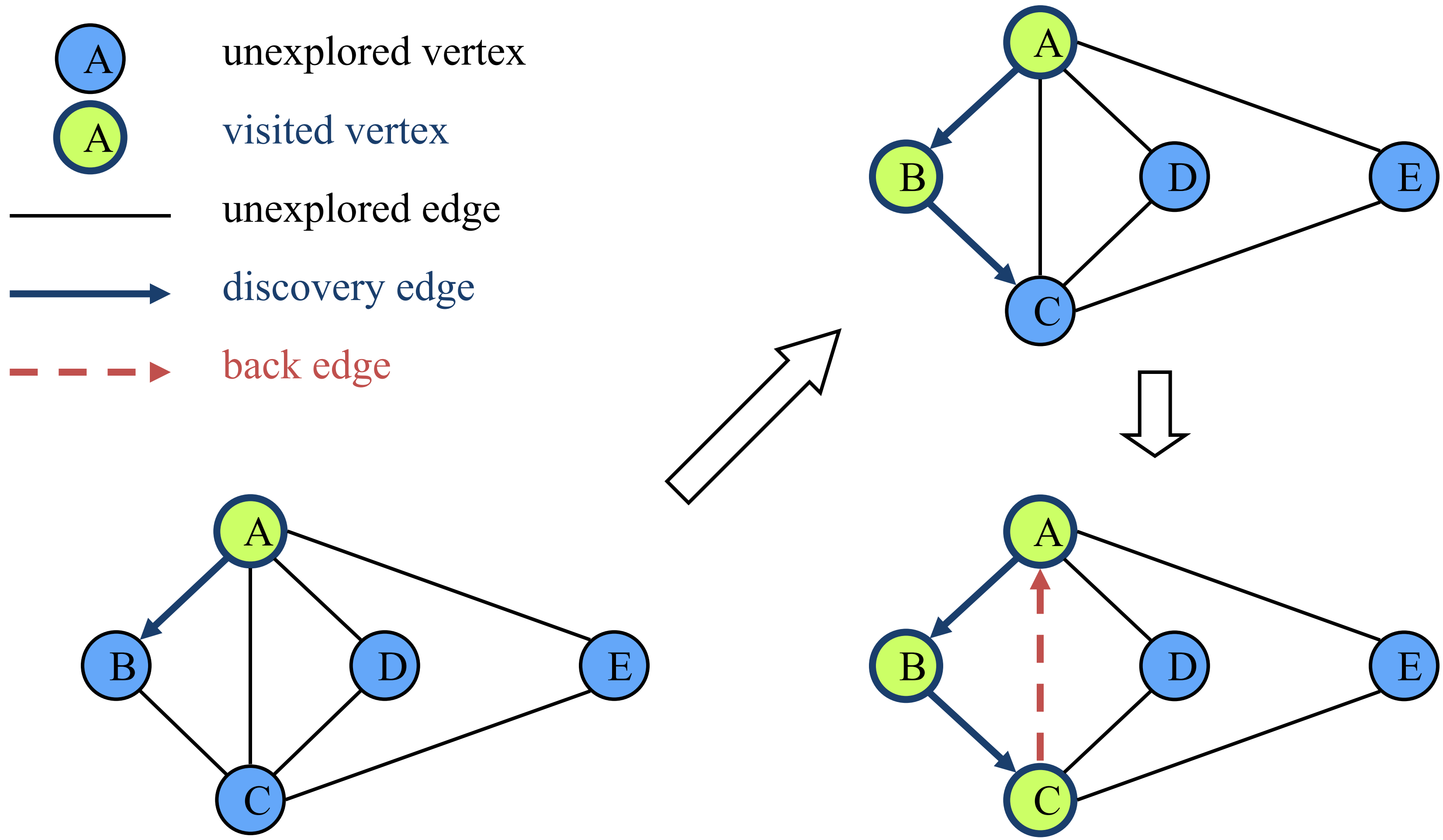
DFS Algorithm

The algorithm uses a mechanism for setting and getting "labels" of vertices and edges.

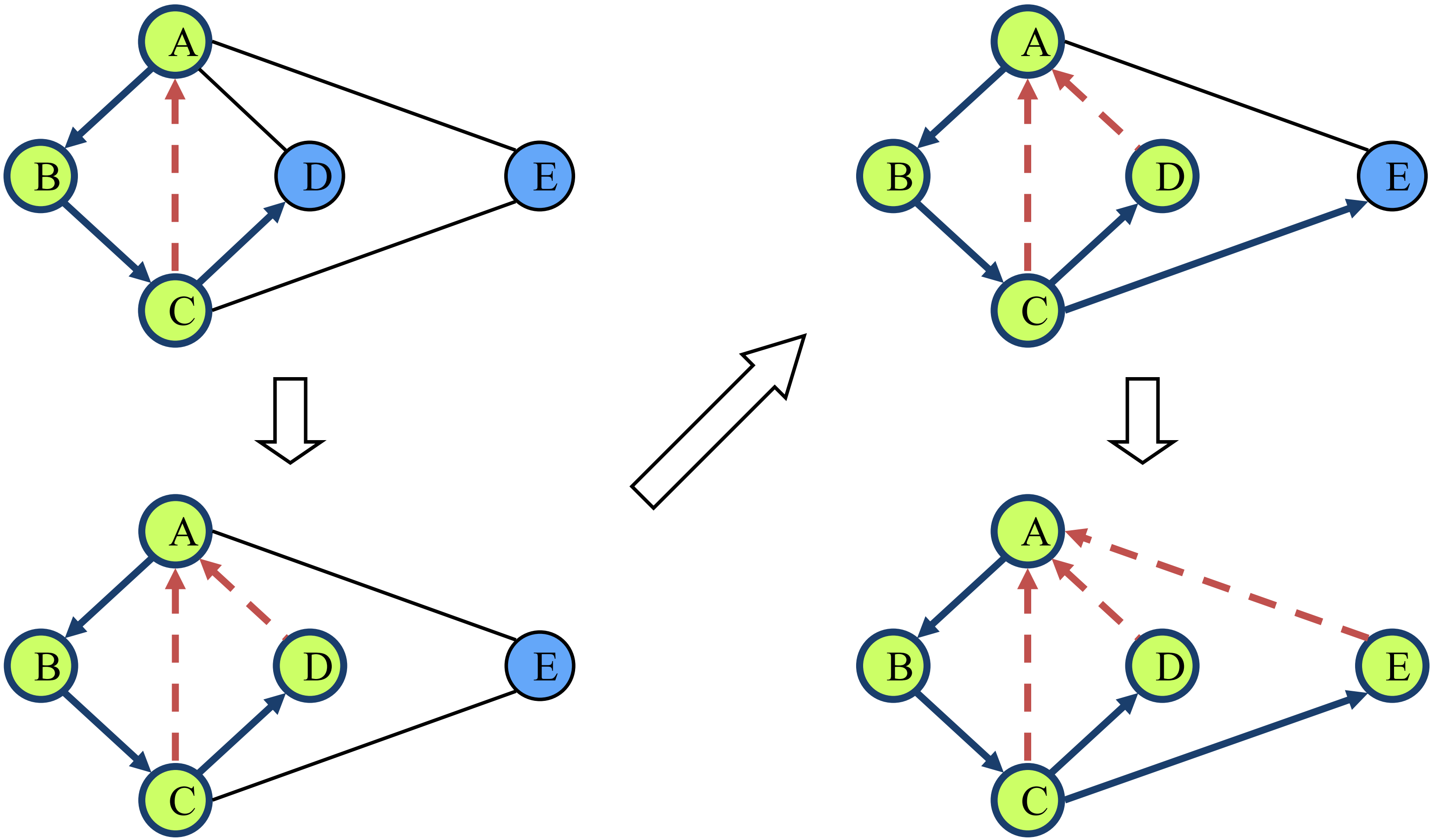
```
Algorithm DFS(G)
    Input graph G
    Output labeling of the edges of G
        as discovery edges and
        back edges
  for all u ∈ G.vertices()
    setLabel(u, UNEXPLORED)
  for all e ∈ G.edges()
    setLabel(e, UNEXPLORED)
  for all v ∈ G.vertices()
    if getLabel(v) = UNEXPLORED
      DFS(G, v)
```

```
Algorithm DFS(G, v)
  Input graph G and a start vertex v of G
  Output labeling of the edges of G
    in the connected component of v
    as discovery edges and back edges
  setLabel(v, VISITED)
  for all e ∈ G.incidentEdges(v)
    if getLabel(e) = UNEXPLORED
      w ← G.opposite(v,e)
      if getLabel(w) = UNEXPLORED
        setLabel(e, DISCOVERY)
        DFS(G, w)
      else
        setLabel(e, BACK)
```

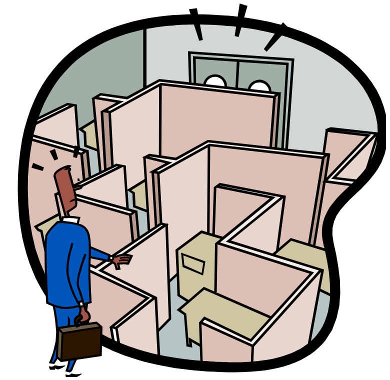
Example


A
unexplored vertex
A
visited vertex
unexplored edge
discovery edge
back edge
A
B
C
D
E
A
B
C
D
E
A
B
C
D
E

Example (cont.)

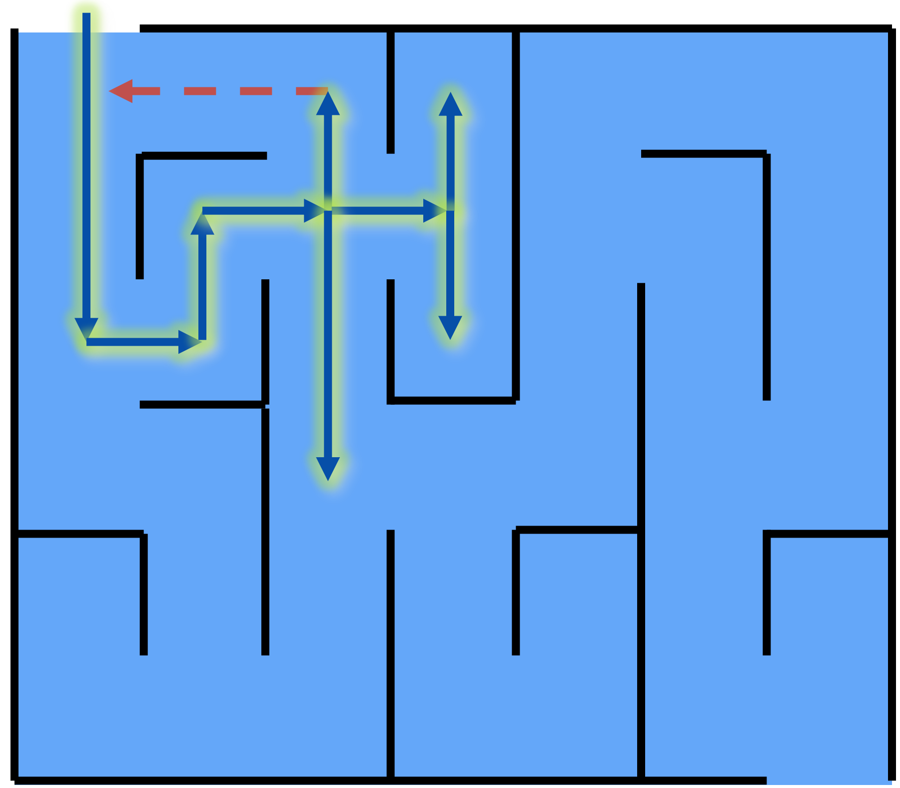


DFS and Maze Traversal



The DFS algorithm is similar to a classic strategy for exploring a maze

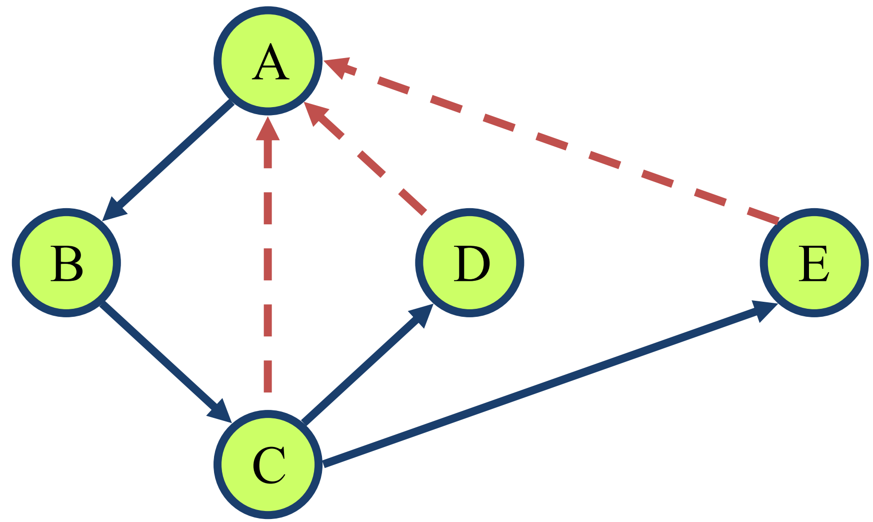
- We mark each intersection, corner and dead end (vertex) visited
- We mark each corridor (edge) traversed
- We keep track of the path back to the entrance (start vertex) by means of a rope (recursion stack)



Properties of DFS

Property 1

DFS*(*G*, *v*)** visits all the vertices and edges in the connected component of ***v



Property 2

The discovery edges labeled by ***DFS*(*G*, *v*)** form a spanning tree of the connected component of ***v***

Analysis of DFS

- Setting/getting a vertex/edge label takes $\boldsymbol{O}(1)$ time
- Each vertex is labeled twice
  - once as UNEXPLORED
  - once as VISITED
- Each edge is labeled twice
  - once as UNEXPLORED
  - once as DISCOVERY or BACK
- Method incidentEdges is called once for each vertex
- DFS runs in $\boldsymbol{O}(\boldsymbol{n} + \boldsymbol{m})$ time provided the graph is represented by the adjacency list structure
  - Recall that $\Sigma_v \deg(v) = 2m$

Path Finding

- We can specialize the DFS algorithm to find a path between two given vertices ***u*** and *z* using the template method pattern
- We call ***DFS*(*G*, *u*)** with ***u*** as the start vertex
- We use a stack ***S*** to keep track of the path between the start vertex and the current vertex
- As soon as destination vertex *z* is encountered, we return the path as the contents of the stack

```
Algorithm pathDFS(G, v, z)
  setLabel(v, VISITED)
  S.push(v)
  if v = z
    return S.elements()
  for all e ∈ G.incidentEdges(v)
    if getLabel(e) = UNEXPLORED
      w ← opposite(v, e)
      if getLabel(w) = UNEXPLORED
        setLabel(e, DISCOVERY)
        S.push(e)
        pathDFS(G, w, z)
        S.pop()        { e gets popped }
      else
        setLabel(e, BACK)
  S.pop()              { v gets popped }
```

Cycle Finding

- We can specialize the DFS algorithm to find a simple cycle using the template method pattern
- We use a stack ***S*** to keep track of the path between the start vertex and the current vertex
- As soon as a back edge (***v, w***) is encountered, we return the cycle as the portion of the stack from the top to vertex ***w***

```
Algorithm cycleDFS(G, v)
    setLabel(v, VISITED)
    S.push(v)
    for all  e ∈ G.incidentEdges(v)
        if  getLabel(e) = UNEXPLORED
            w ← opposite(v,e)
            S.push(e)
            if getLabel(w) = UNEXPLORED
                setLabel(e, DISCOVERY)
                cycleDFS(G, w)
                S.pop()
            else
                C ← new empty stack
                repeat
                    o ← S.pop()
                    C.push(o)
                until o = w
                return C.elements()
    S.pop()
```