Selection

Selection Problem

- Given an integer k and n elements x_1, x_2, …, x_n, taken from a total order, find the k-th smallest element in this set.

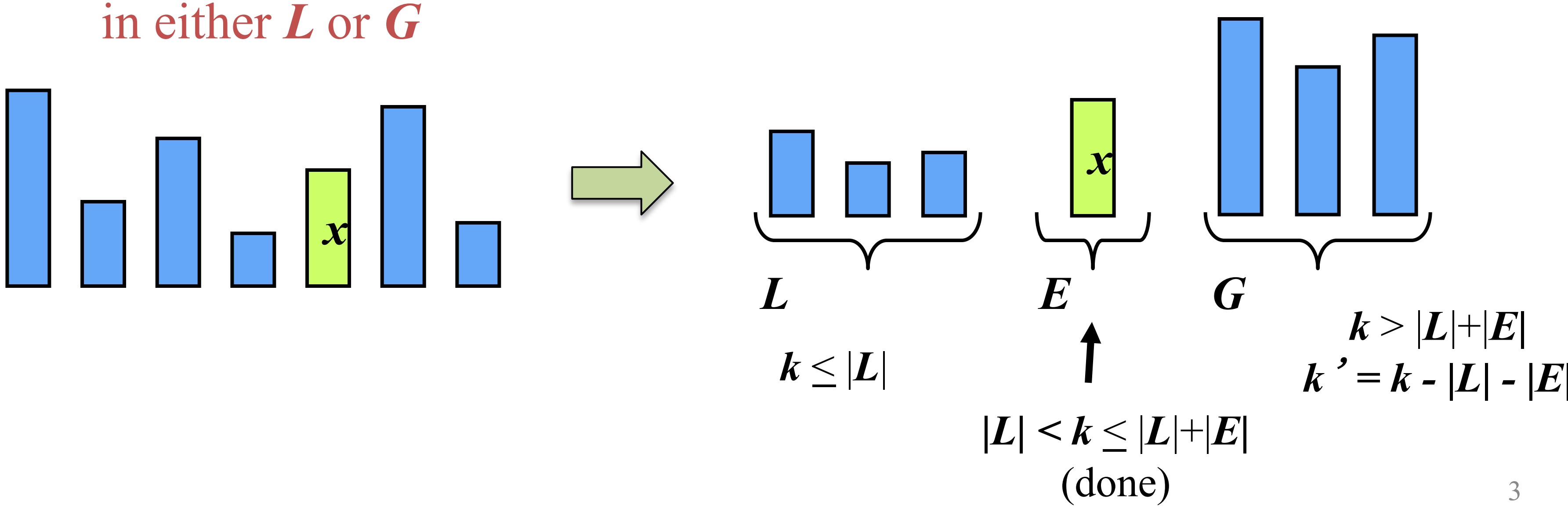
- Of course, we can sort the set in $O(n \log n)$ time and then index the k-th element.
  - Ex when k=3:
    5, 10, 6, 3, 14, 12, 2 → 2, 3, 5, 6, 10, 12, 14

- Can we solve the selection problem faster?

Quick-Select

A randomized selection algorithm based on the prune-and-search paradigm:

- Prune: pick a random element $\boldsymbol{x}$ (called pivot) and partition $\boldsymbol{S}$ into
  - $\boldsymbol{L}$ elements less than $\boldsymbol{x}$
  - $\boldsymbol{E}$ elements equal $\boldsymbol{x}$
  - $\boldsymbol{G}$ elements greater than $\boldsymbol{x}$
- Search: depending on k, either answer is in $\boldsymbol{E}$, or we need to recurse in either $\boldsymbol{L}$ or $\boldsymbol{G}$



Partition

We partition an input sequence as in the quick-sort algorithm:

- Remove, in turn, each element ***y*** from ***S*** and
- Insert ***y*** into ***L***, ***E*** or ***G***, depending on the result of the comparison with the pivot ***p***

Each insertion and removal takes $O(1)$ time

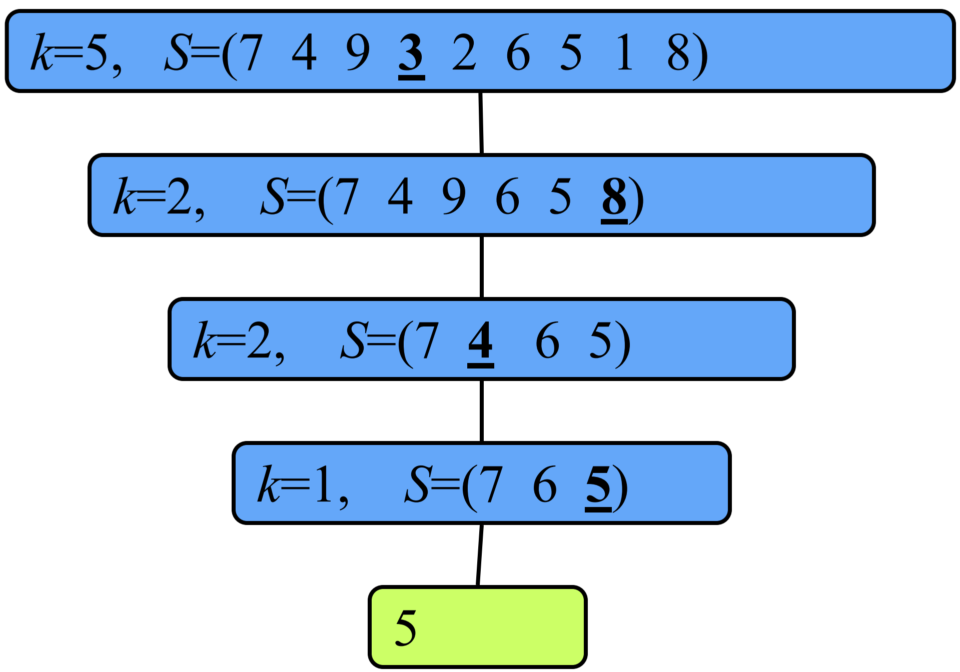
Thus, the partition step of quick-select takes $O(n)$ time

Algorithm *partition*(*S*, *p*)
  Input sequence *S*, pivot *p*
  Output subsequences *L*, *E*, *G* of the elements of *S* less than, equal to, or greater than the pivot, resp.
  L, *E*, *G* ← empty sequences
  while ¬*S.isEmpty*()
    y ← *S.remove*(*S.first*())
    if $y < p$
      L.insertLast(*y*)
    else if $y = p$
      E.insertLast(*y*)
    else { $y > p$ }
      G.insertLast(*y*)
  return *L*, *E*, *G*

Quick-Select Visualization

An execution of quick-select can be visualized by a recursion path

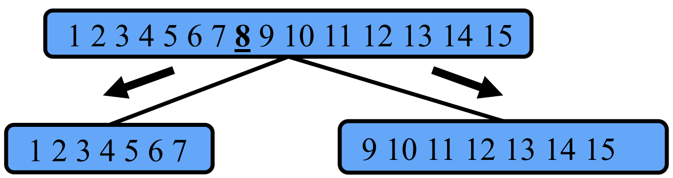
- each node represents a recursive call of quick-select, and stores k and the remaining sequence



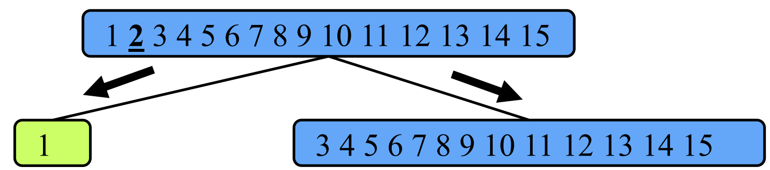
Expected Running Time

Consider a recursive call of quick-select on a sequence of size s

- **Good call:** the sizes of L and G are each less than $3s/4$
- **Bad call:** one of L and G has size greater than $3s/4$



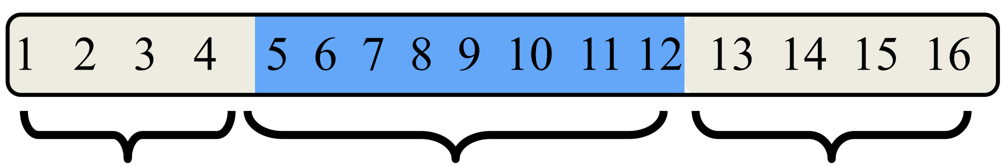
Good call



Bad call

A call is good with probability 1/2

- 1/2 of the possible pivots cause good calls:



Bad pivots **Good pivots** **Bad pivots**

Expected Running Time (2)

Probabilistic Fact #1: The expected number of coin tosses required in order to get one head is two.

Probabilistic Fact #2: Expectation is a linear function:

- $E(X + Y) = E(X) + E(Y)$
- $E(cX) = cE(X)$

Let $T(n)$ denote the expected running time of quick-select.

- By Fact #2,
  - $T(n) \leq T(3n/4) + bn$*(expected # of calls before a good call)
- By Fact #1,
  - $T(n) \leq T(3n/4) + 2bn$
- That is, $T(n)$ is a geometric series:
  - $T(n) \leq 2bn + 2b(3/4)n + 2b(3/4)^2 n + 2b(3/4)^3 n + \ldots$
- So $T(n)$ is $O(n)$.

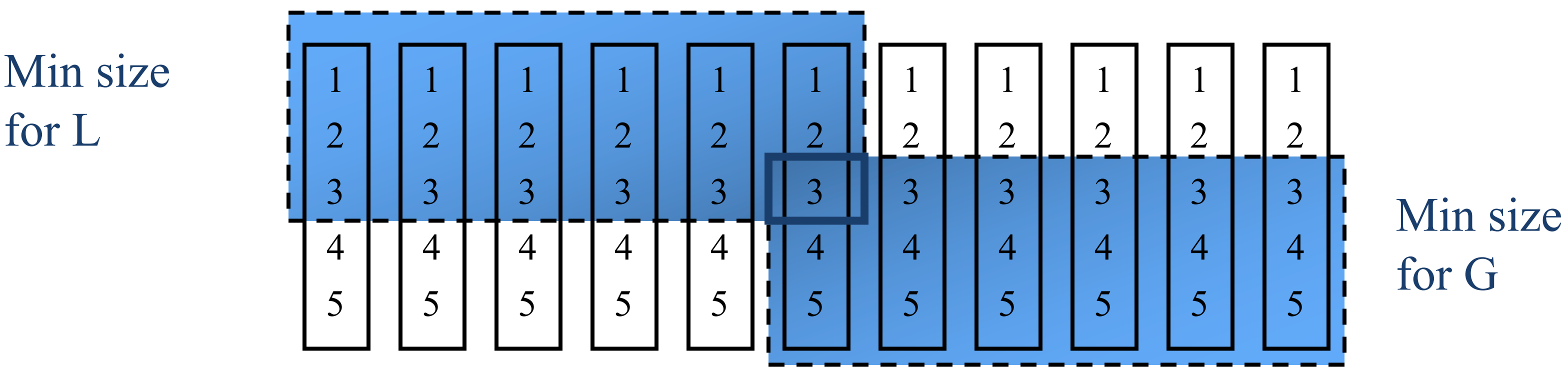
Randomized quick-select runs in $O(n)$ expected time.

Deterministic Selection

We *can* do selection in $O(n)$ worst-case time.

Main idea: recursively use the selection algorithm itself to find a good pivot for quick-select

- Divide S into $n/5$ sets of 5 each
- Find a median in each set
- Recursively find the median of the "baby" medians.
- Use median of medians as a guaranteed good pivot



See Exercise C-4.24 for details of analysis.