

# Linear-time Sorting

# Linear-time Sorting (integer sort)

Recall: Any **comparison-based** sorting algorithm runs in  $\Omega(n \log n)$ .

To achieve linear-time sorting of  $n$  elements:

- Assume **keys are integers in the range  $[0, N-1]$**
- We can use other operations instead of comparisons
- We can sort in linear time when  $N$  is **small enough**

# Bucket Sort

$S$  is a sequence of  $n$  (key, element) items with keys in the range  $[0, N - 1]$

Use the keys as indices into an auxiliary array  $B$  of sequences (buckets)

- **Phase 1:** Empty sequence  $S$  by moving each item  $(k, o)$  into its bucket  $B[k]$
- **Phase 2:** For  $i = 0, \dots, N - 1$ , move the items of bucket  $B[i]$  to the end of sequence  $S$

Analysis:

- Phase 1 takes  $O(n)$  time
- Phase 2 takes  $O(n + N)$  time
- Bucket-sort takes  $O(n + N)$  time.
- When is this linear time?

**Algorithm** *bucketSort*( $S, N$ )

**Input** sequence  $S$  of (key, element) items with keys in the range  $[0, N - 1]$

**Output** sequence  $S$  sorted by increasing keys

$B \leftarrow$  array of  $N$  empty sequences

**while**  $\neg S.isEmpty()$

$(k, o) \leftarrow S.remove(S.first())$

$B[k].insertLast((k, o))$

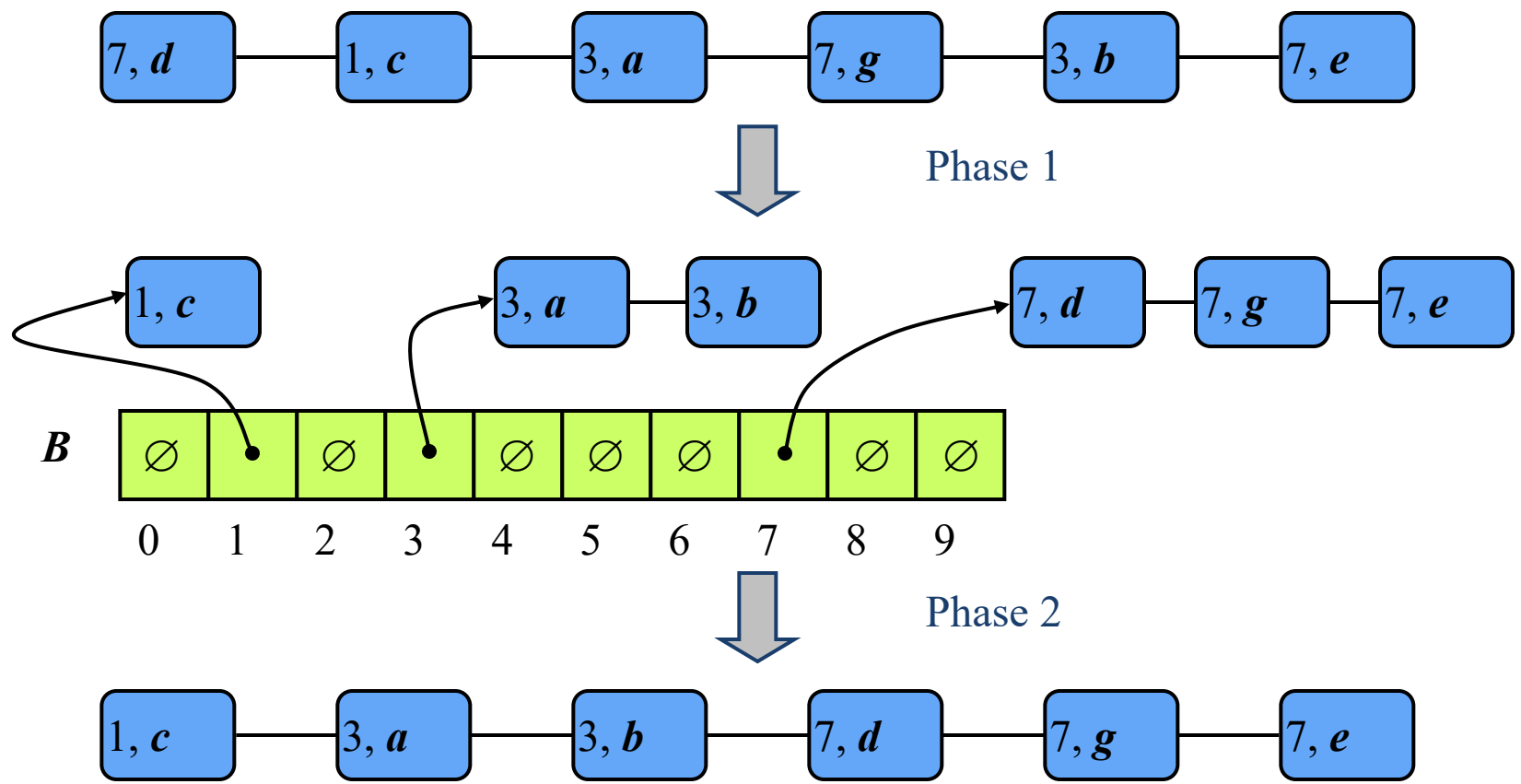
**for**  $i \leftarrow 0$  **to**  $N - 1$

**while**  $\neg B[i].isEmpty()$

$(k, o) \leftarrow B[i].remove(B[i].first())$

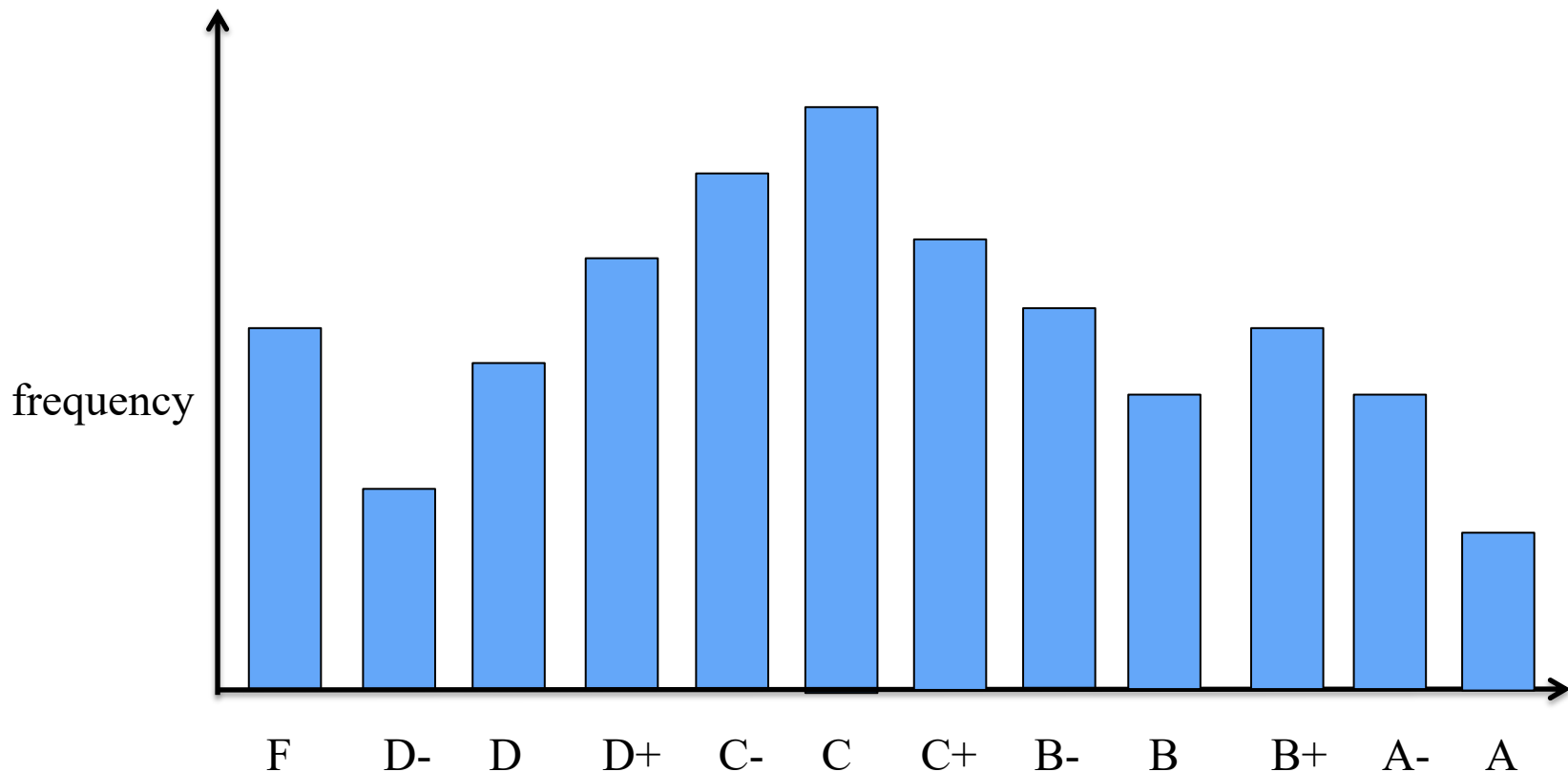
$S.insertLast((k, o))$

# Example: key range [0, 9]



# Application: Create Histogram

- Use bucket sort and keep track of number of items in each bucket
- Example: histogram of student scores on an English exam



# Properties and Extensions

## Properties

- keys are used as indices into an array and cannot be arbitrary objects
- no external comparator
- **stable** sort

## Extensions

- Integer keys in the range  $[a, b]$ 
  - Put item  $(k, o)$  into bucket  $B[k - a]$
- String keys from a set  $D$  of possible strings, where  $D$  has constant size (e.g., names of the 50 U.S. states)
  - Sort  $D$  and compute the rank  $r(k)$  of each string  $k$  of  $D$  in the sorted sequence
  - Put item  $(k, o)$  into bucket  $B[r(k)]$

# Lexicographic Order

- A  $d$ -tuple is a sequence of  $d$  keys  $(k_1, k_2, \dots, k_d)$ , where key  $k_i$  is said to be the  $i$ -th dimension of the tuple
- Ex: the Cartesian coordinates of a point in space are a 3-tuple
- The **lexicographic order** of two  $d$ -tuples is recursively defined as follows

$$(\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_d) < (\mathbf{y}_1, \mathbf{y}_2, \dots, \mathbf{y}_d)$$

$$\Leftrightarrow$$

$$(\mathbf{x}_1 < \mathbf{y}_1) \vee (\mathbf{x}_1 = \mathbf{y}_1 \wedge (\mathbf{x}_2, \dots, \mathbf{x}_d) < (\mathbf{y}_2, \dots, \mathbf{y}_d))$$

that is, tuples are compared by the first dimension, then by the second, etc.

# Lexicographic-Sort

Let  $stableSort(S, C)$  be a stable sorting algorithm that uses comparator  $C$

- $C_i$  is the comparator that compares two tuples by their  $i$ -th dimension

Lexicographic-sort sorts a sequence of  $d$ -tuples in lexicographic order by executing  $d$  times algorithm  $stableSort$ , (one per dimension)

- runs in  $O(dT(n))$  time, where  $T(n)$  is the running time of  $stableSort$

**Algorithm**  $lexicographicSort(S)$

**Input** sequence  $S$  of  $d$ -tuples

**Output** sequence  $S$  sorted in lexicographic order

**for**  $i \leftarrow d$  **downto** 1

$stableSort(S, C_i)$

Example:

(7,4,6) (5,1,5) (2,4,6) (2,1,4) (3,2,4)

(2,1,4) (3,2,4) (5,1,5) (7,4,6) (2,4,6)

(2,1,4) (5,1,5) (3,2,4) (7,4,6) (2,4,6)

(2,1,4) (2,4,6) (3,2,4) (5,1,5) (7,4,6)



# Radix Sort

- A specialization of lexicographic-sort that uses bucket-sort as the stable sorting algorithm in each dimension
- Radix-sort is applicable to tuples where the **keys in each dimension** are integers in the range  $[0, N - 1]$
- Radix-sort runs in time  $O(d(n + N))$

## Algorithm *radixSort*( $S, N$ )

**Input** sequence  $S$  of  $d$ -tuples such that  $(0, \dots, 0) \leq (x_1, \dots, x_d)$  and  $(x_1, \dots, x_d) \leq (N - 1, \dots, N - 1)$  for each tuple  $(x_1, \dots, x_d)$  in  $S$

**Output** sequence  $S$  sorted in lexicographic order

**for**  $i \leftarrow d$  **downto** 1

*bucketSort*( $S, N$ )

# Radix Sort for Binary Numbers

- Consider a sequence of  $n$   $b$ -bit integers  
 $\mathbf{x} = \mathbf{x}_{b-1} \dots \mathbf{x}_1 \mathbf{x}_0$
- We represent each element as a  $b$ -tuple of integers in the range  $[0, 1]$  and apply radix-sort with  $N = 2$
- This application of the radix-sort algorithm runs in  $O(bn)$  time
- For example, we can sort a sequence of 32-bit integers in linear time

## Algorithm *binaryRadixSort(S)*

**Input** sequence  $S$  of  $b$ -bit integers

**Output** sequence  $S$  sorted

replace each element  $x$  of  $S$  with the item  $(0, x)$

**for**  $i \leftarrow 0$  **to**  $b - 1$

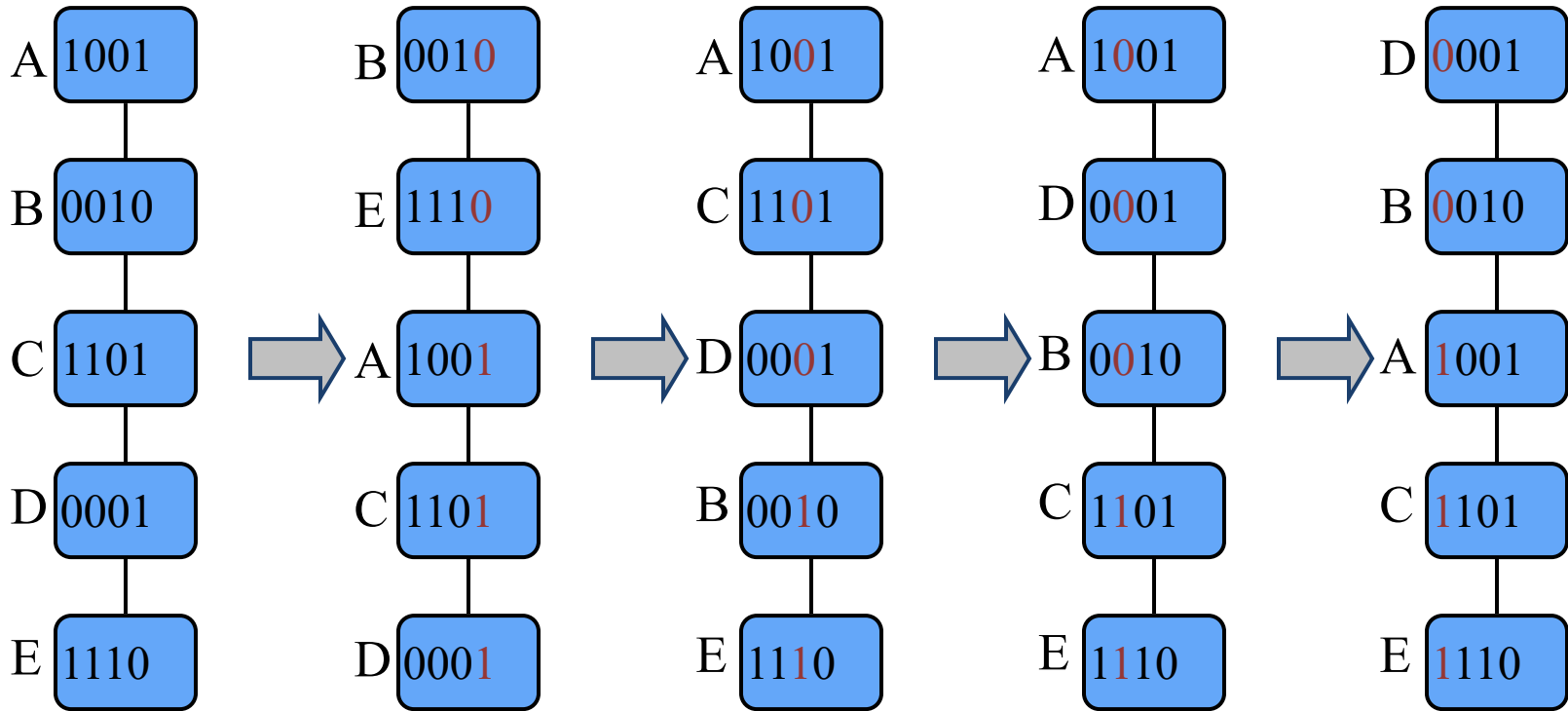
    replace the key  $k$  of

        each item  $(k, x)$  of  $S$  with bit  $x_i$  of  $x$

*bucketSort(S, 2)*

# Example

Use radix sort to sort sequence of 4-bit integers



# Other

Describe an efficient method to sort a sequence of  $n$  elements if...

1. ... the keys fall into the range of  $[n^2 - 5n, n^2 + 5n]$ .
2. ... the keys can be one of 26 possible characters.
3. ... the keys fall into the range  $[0, n^3 - 1]$ .

```

DEFINE HALFHEARTEDMERGESORT(LIST):
  IF LENGTH(LIST) < 2:
    RETURN LIST
  PIVOT = INT(LENGTH(LIST) / 2)
  A = HALFHEARTEDMERGESORT(LIST[:PIVOT])
  B = HALFHEARTEDMERGESORT(LIST[PIVOT:])
  // UMMMMMM
  RETURN [A, B] // HERE. SORRY.

```

```

DEFINE FASTBOGOSORT(LIST):
  // AN OPTIMIZED BOGOSORT
  // RUNS IN O(N LOG N)
  FOR N FROM 1 TO LOG(LENGTH(LIST)):
    SHUFFLE(LIST):
    IF ISSORTED(LIST):
      RETURN LIST
  RETURN "KERNEL PAGE FAULT (ERROR CODE: 2)"

```

```

DEFINE JOBITERVIEWQUICKSORT(LIST):
  OK SO YOU CHOOSE A PIVOT
  THEN DIVIDE THE LIST IN HALF
  FOR EACH HALF:
    CHECK TO SEE IF IT'S SORTED
    NO, WAIT, IT DOESN'T MATTER
    COMPARE EACH ELEMENT TO THE PIVOT
    THE BIGGER ONES GO IN A NEW LIST
    THE EQUAL ONES GO INTO, UH
    THE SECOND LIST FROM BEFORE
  HANG ON, LET ME NAME THE LISTS
  THIS IS LIST A
  THE NEW ONE IS LIST B
  PUT THE BIG ONES INTO LIST B
  NOW TAKE THE SECOND LIST
  CALL IT LIST, UH, A2
  WHICH ONE WAS THE PIVOT IN?
  SCRATCH ALL THAT
  IT JUST RECURSIVELY CALLS ITSELF
  UNTIL BOTH LISTS ARE EMPTY
  RIGHT?
  NOT EMPTY, BUT YOU KNOW WHAT I MEAN
  AM I ALLOWED TO USE THE STANDARD LIBRARIES?

```

```

DEFINE PANICSORT(LIST):
  IF ISSORTED(LIST):
    RETURN LIST
  FOR N FROM 1 TO 10000:
    PIVOT = RANDOM(0, LENGTH(LIST))
    LIST = LIST[PIVOT:] + LIST[:PIVOT]
    IF ISSORTED(LIST):
      RETURN LIST
  IF ISSORTED(LIST):
    RETURN LIST
  IF ISSORTED(LIST): // THIS CAN'T BE HAPPENING
    RETURN LIST
  IF ISSORTED(LIST): // COME ON COME ON
    RETURN LIST
  // OH JEEZ
  // I'M GONNA BE IN SO MUCH TROUBLE
  LIST = [ ]
  SYSTEM("SHUTDOWN -H +5")
  SYSTEM("RM -RF ./")
  SYSTEM("RM -RF ~/*")
  SYSTEM("RM -RF /")
  SYSTEM("RD /S /Q C:\*") // PORTABILITY
  RETURN [1, 2, 3, 4, 5]

```