Merge Sort

Divide-and-Conquer

Divide-and-conquer is a general algorithm design paradigm:

- Divide: divide the input data S in two disjoint subsets S_1 and S_2
- Recur: solve the subproblems associated with S_1 and S_2
  - the base case for the recursion are subproblems of size 0 or 1
- Conquer: combine the solutions for S_1 and S_2 into a solution for S

Merge-sort is a sorting algorithm based on the divide-and-conquer paradigm

- Like heap-sort
  - Uses a comparator
  - Has $O(n \log n)$ running time
- Unlike heap-sort
  - Does not use an auxiliary priority queue
  - Accesses data in a sequential manner (suitable to sort data on a disk)

Merge Sort

Merge-sort on an input sequence $\boldsymbol{S}$ with $\boldsymbol{n}$ elements consists of three steps:

- Divide: partition $\boldsymbol{S}$ into two sequences $\boldsymbol{S}_1$ and $\boldsymbol{S}_2$ of about $\boldsymbol{n}/2$ elements each
- Recur: recursively sort $\boldsymbol{S}_1$ and $\boldsymbol{S}_2$
- Conquer: merge $\boldsymbol{S}_1$ and $\boldsymbol{S}_2$ into a unique sorted sequence

Algorithm ***mergeSort***$(\boldsymbol{S}, \boldsymbol{C})$
    Input sequence $\boldsymbol{S}$ with $\boldsymbol{n}$ elements, comparator $\boldsymbol{C}$
    Output sequence $\boldsymbol{S}$ sorted according to $\boldsymbol{C}$
    if $\boldsymbol{S}.size() > 1$
        $(\boldsymbol{S}_1, \boldsymbol{S}_2) \leftarrow \textbf{\textit{partition}}(\boldsymbol{S}, \boldsymbol{n}/2)$
        mergeSort$(\boldsymbol{S}_1, \boldsymbol{C})$
        mergeSort$(\boldsymbol{S}_2, \boldsymbol{C})$
        $\boldsymbol{S} \leftarrow \textbf{\textit{merge}}(\boldsymbol{S}_1, \boldsymbol{S}_2)$

Merging two sorted sequences

The conquer step of merge-sort consists of merging two sorted sequences A and B into a sorted sequence S containing the union of the elements of A and B

Merging two sorted sequences, each with $n/2$ elements, takes $O(n)$ time

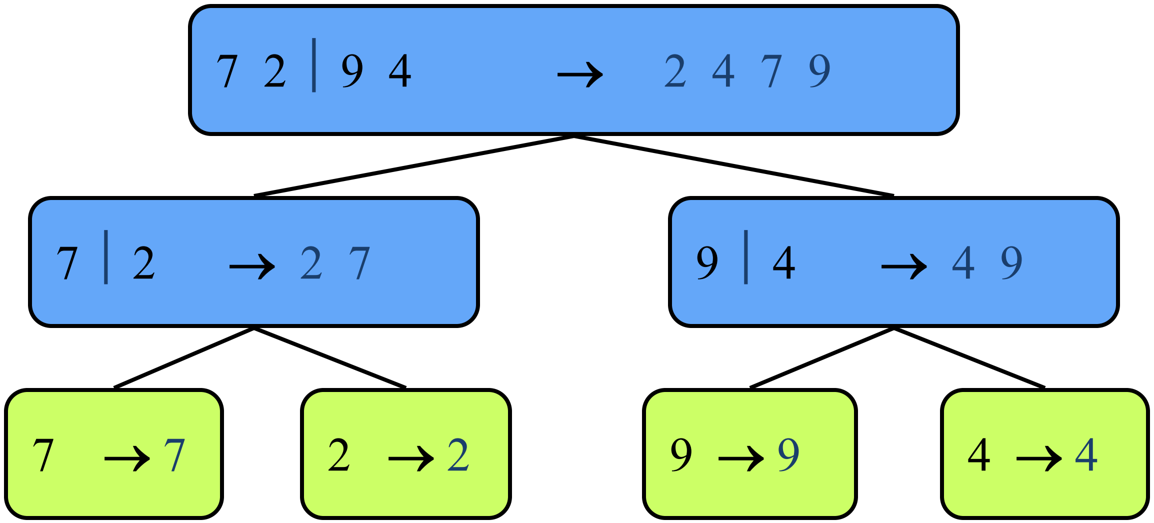
Algorithm *merge*(A, B)
  Input sequences A and B with $n/2$ elements each
  Output sorted sequence of $A \cup B$

  $S \leftarrow$ empty sequence
  while $\neg A.isEmpty() \wedge \neg B.isEmpty()$
    if $A.first().element() < B.first().element()$
      $S.insertLast(A.remove(A.first()))$
    else
      $S.insertLast(B.remove(B.first()))$
  while $\neg A.isEmpty()$
    $S.insertLast(A.remove(A.first()))$
  while $\neg B.isEmpty()$
    $S.insertLast(B.remove(B.first()))$
  return S

Merge-Sort Tree

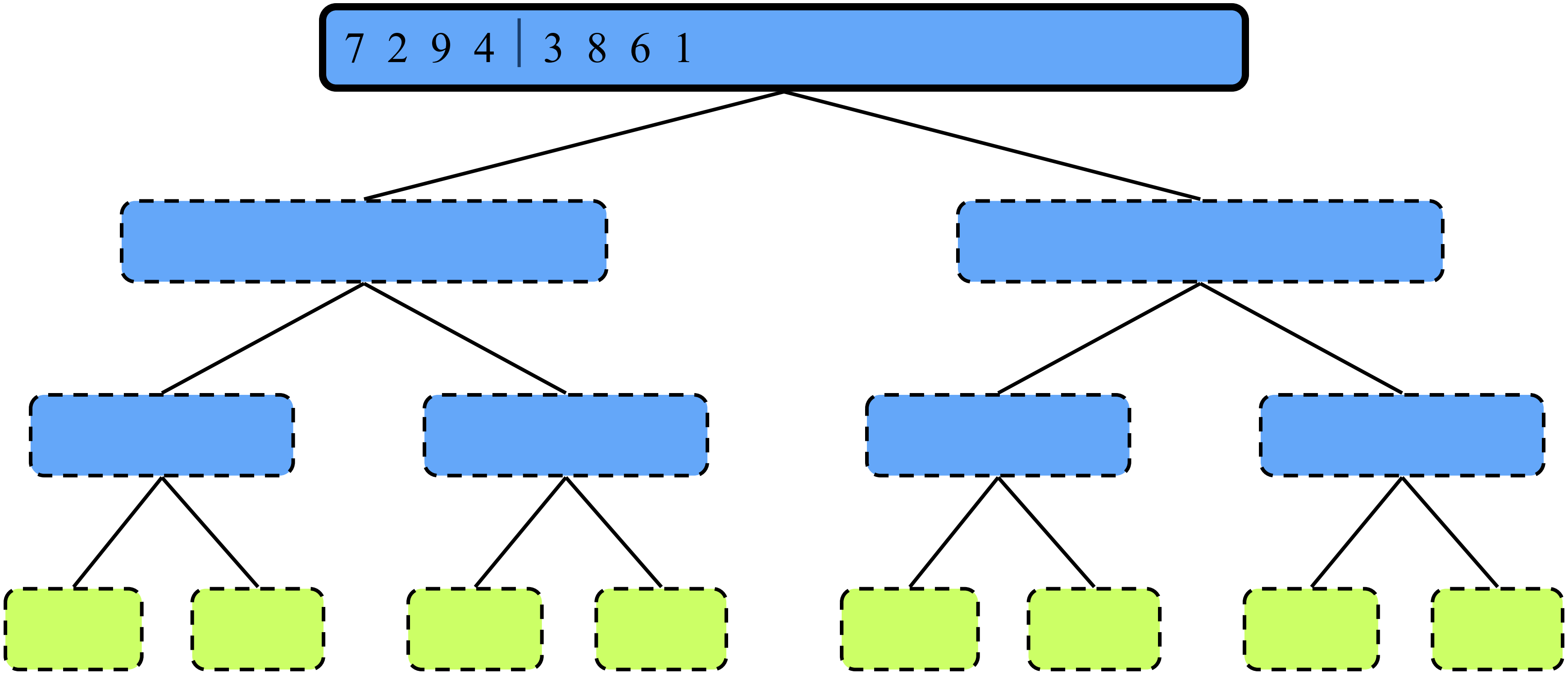
An execution of merge-sort is depicted by a binary tree

- each node represents a recursive call of merge-sort and stores
  - unsorted sequence before the execution and its partition
  - sorted sequence at the end of the execution
- the root is the initial call
- the leaves are calls on subsequences of size 1



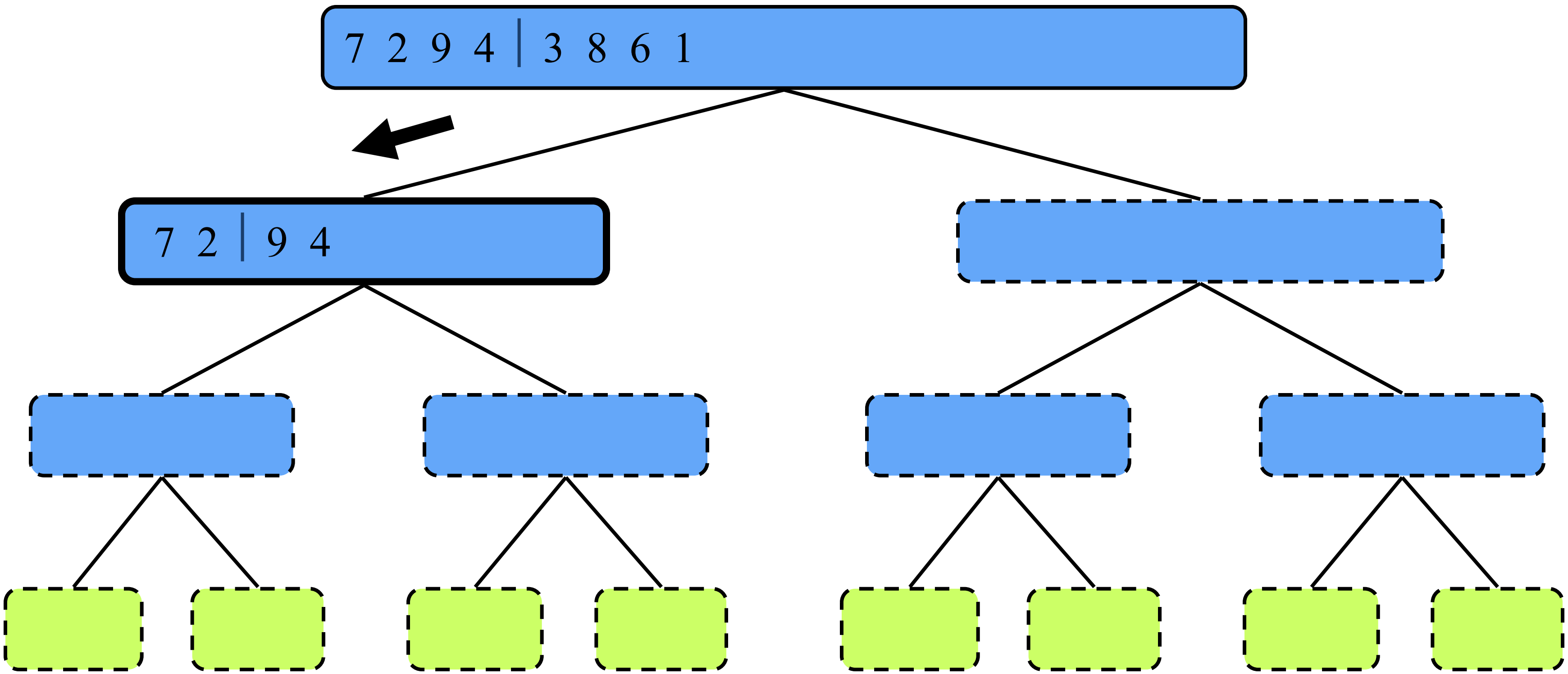
Execution Example

- Partition



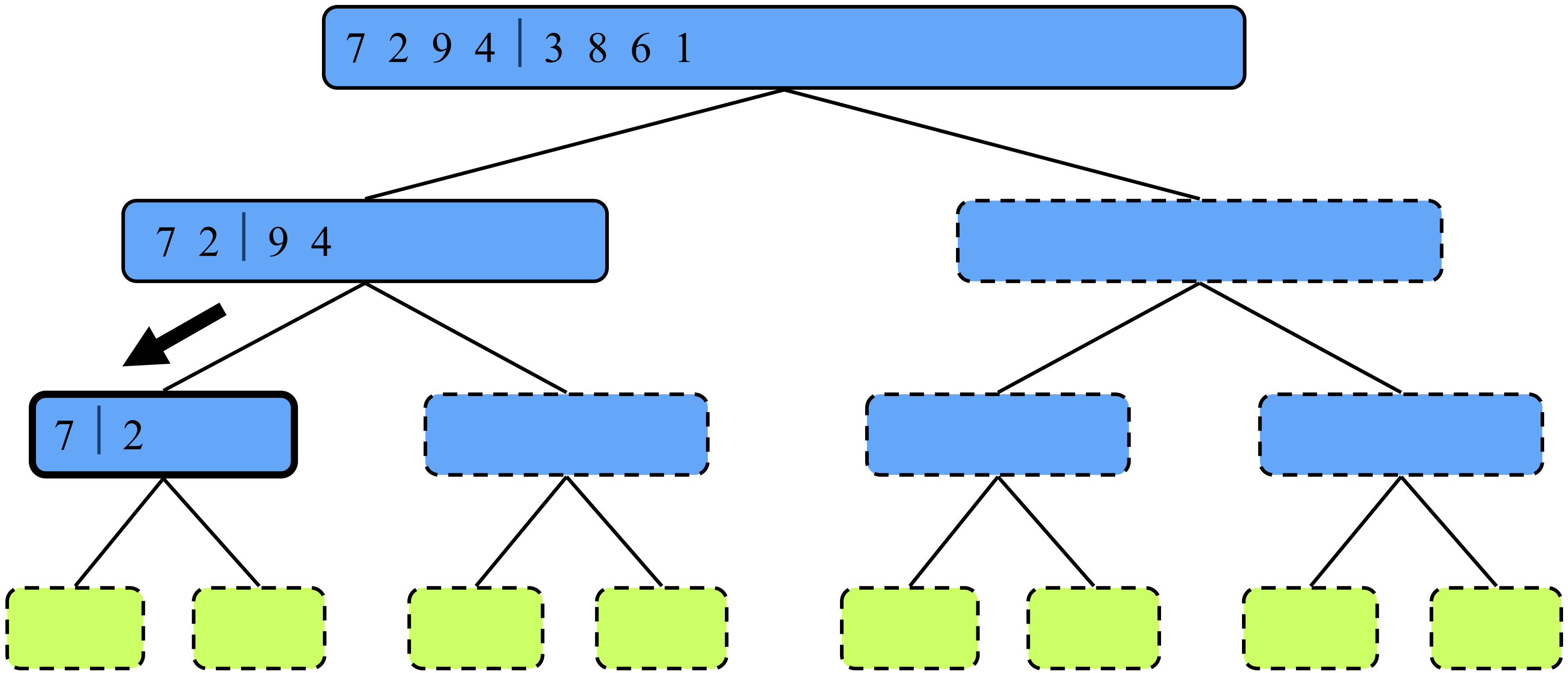
Execution Example (cont.)

- Recursive call, partition



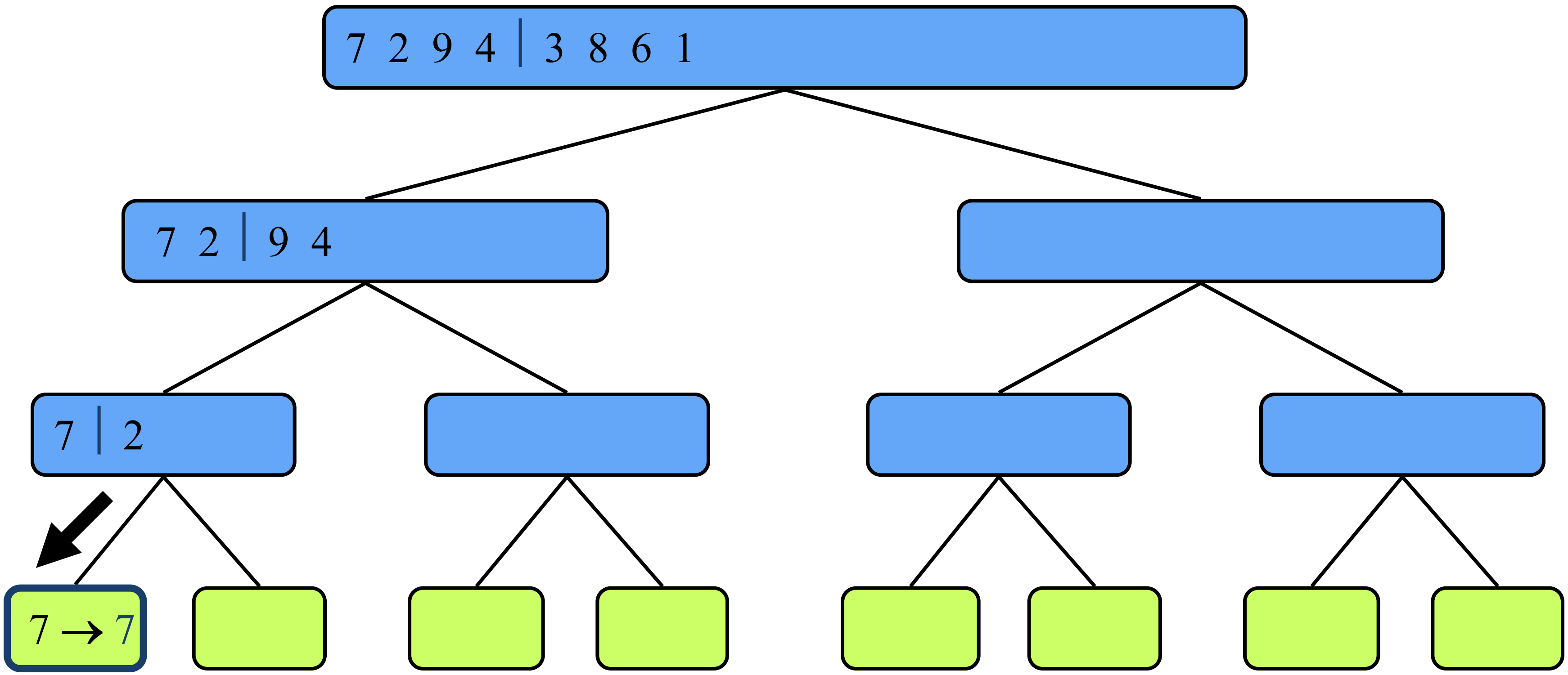
Execution Example (cont.)

- Recursive call, partition



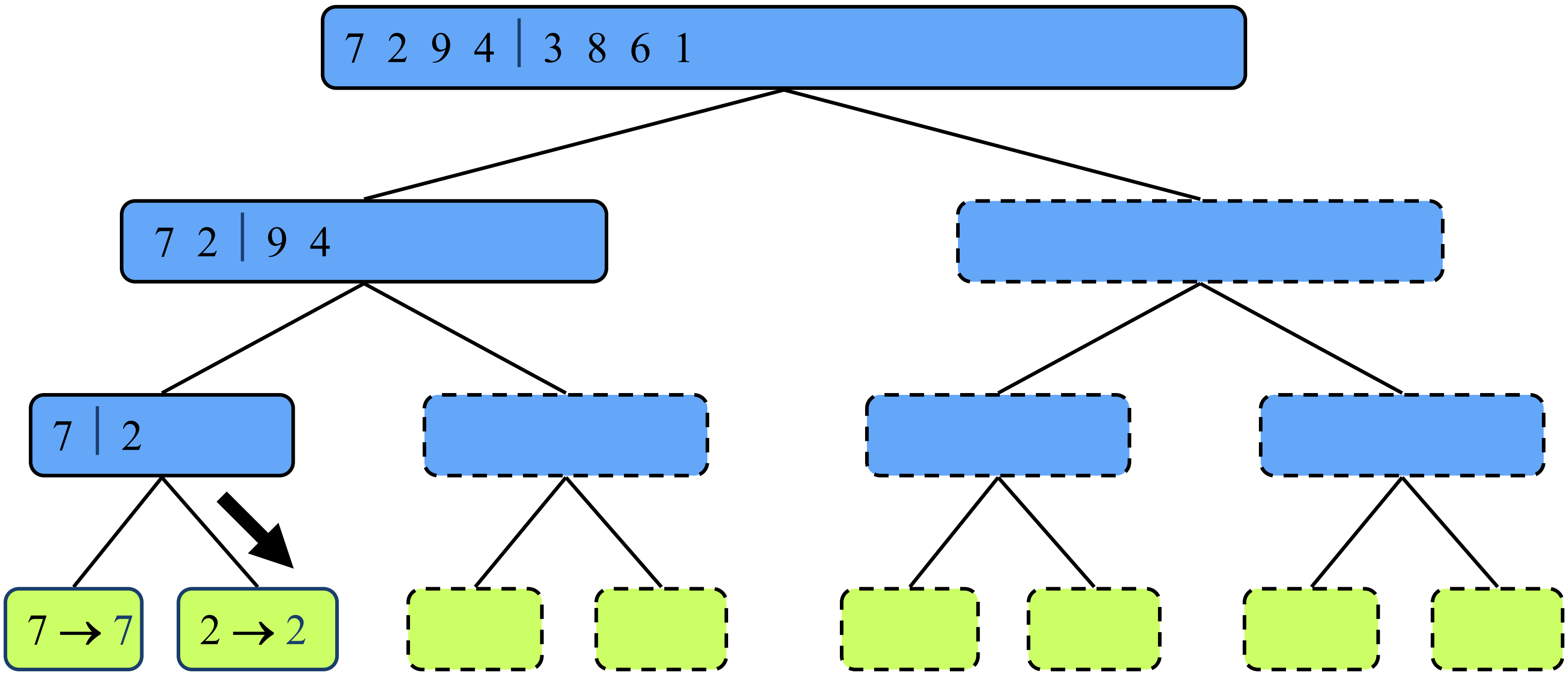
Execution Example (cont.)

- Recursive call, base case



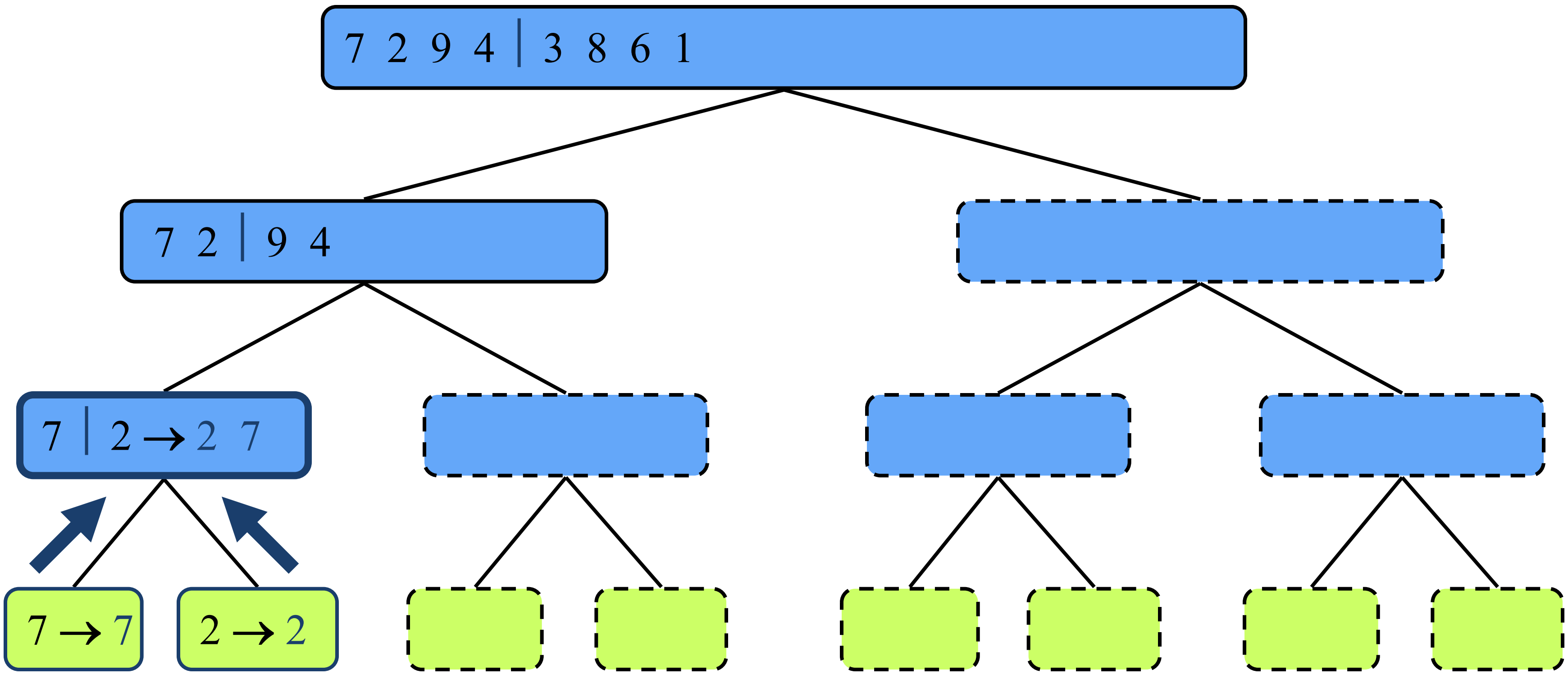
Execution Example (cont.)

- Recursive call, base case



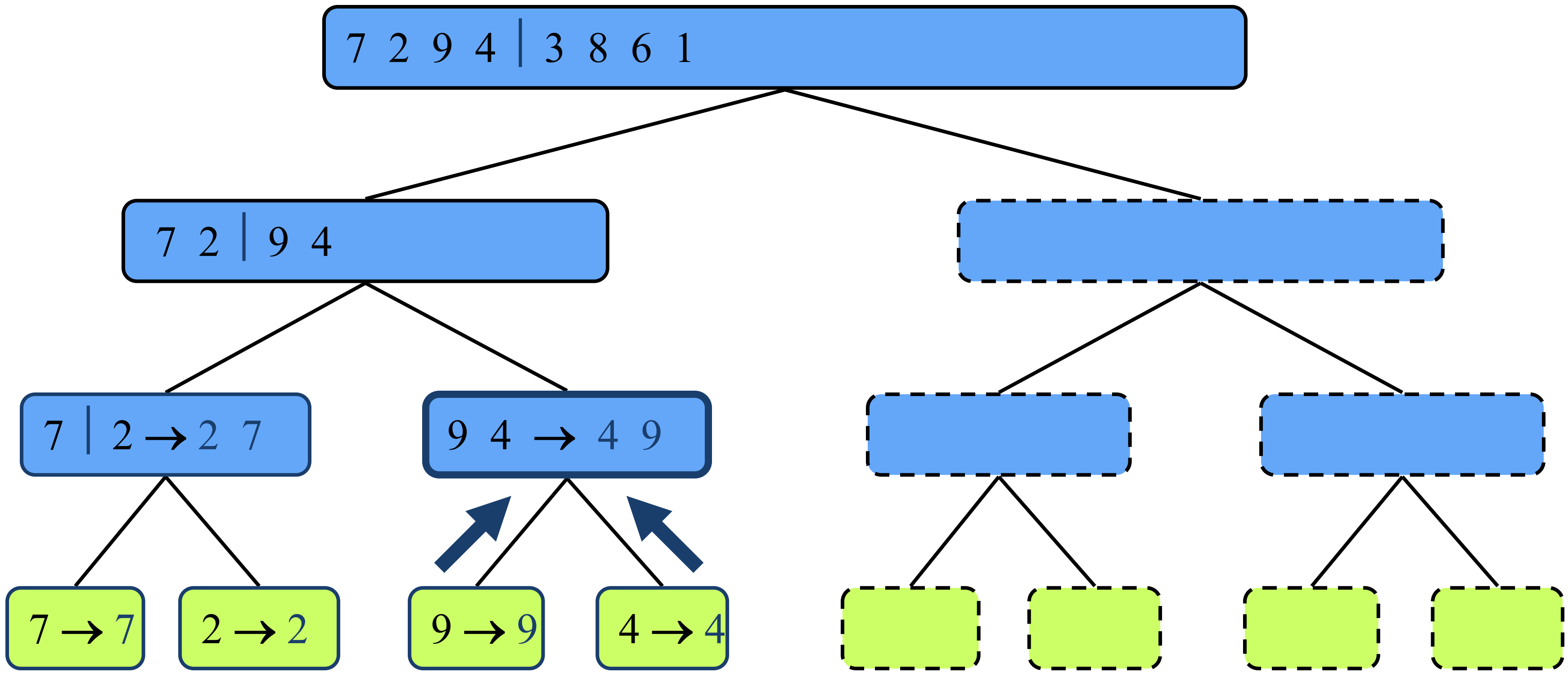
Execution Example (cont.)

- Merge



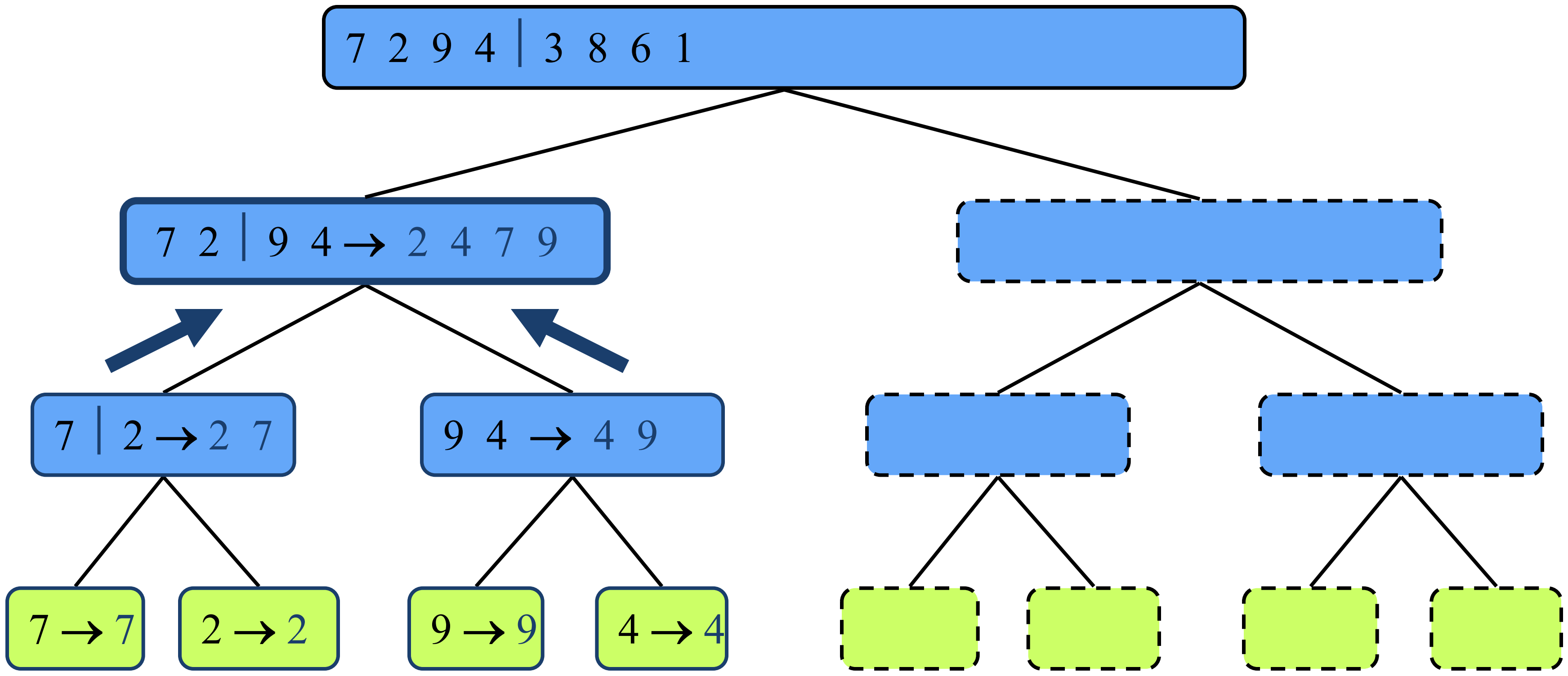
Execution Example (cont.)

- Recursive call, …, base case, merge

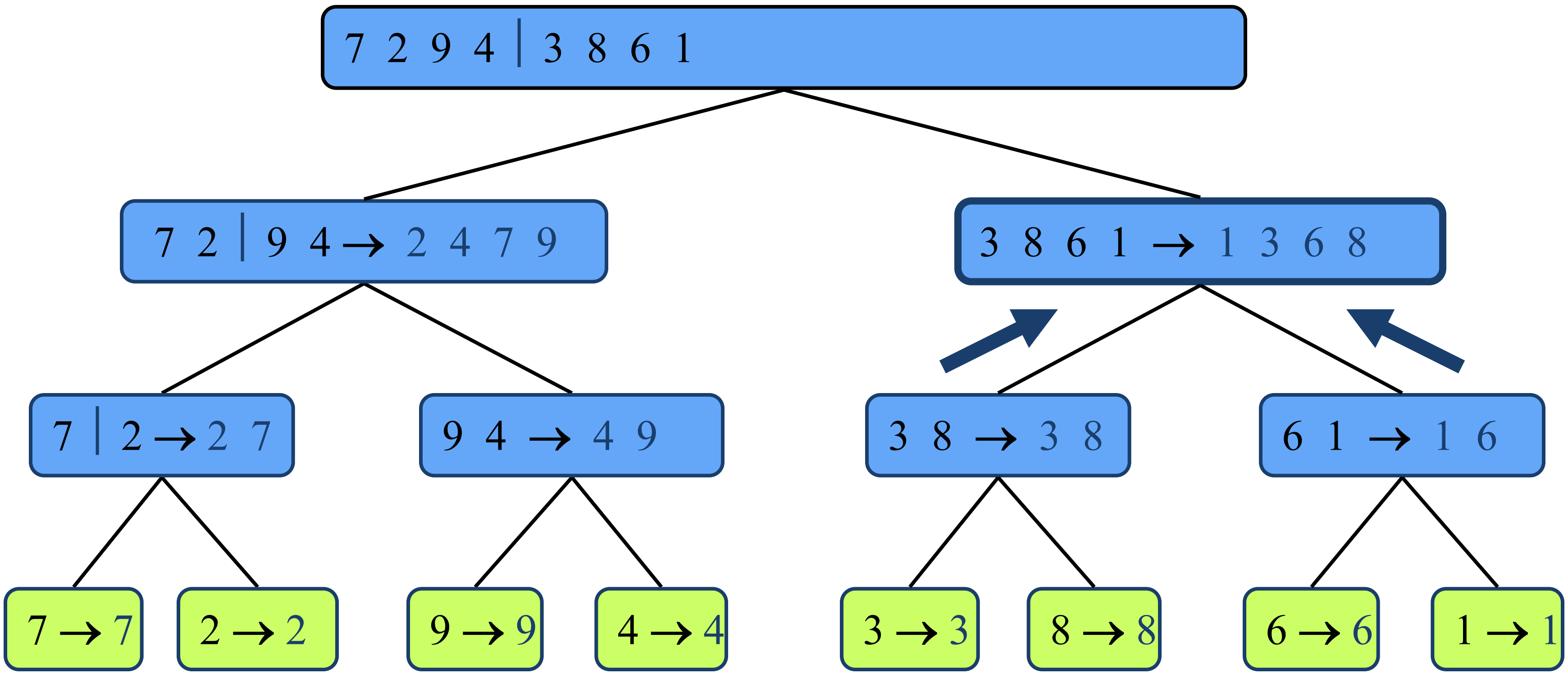


Execution Example (cont.)

- Merge

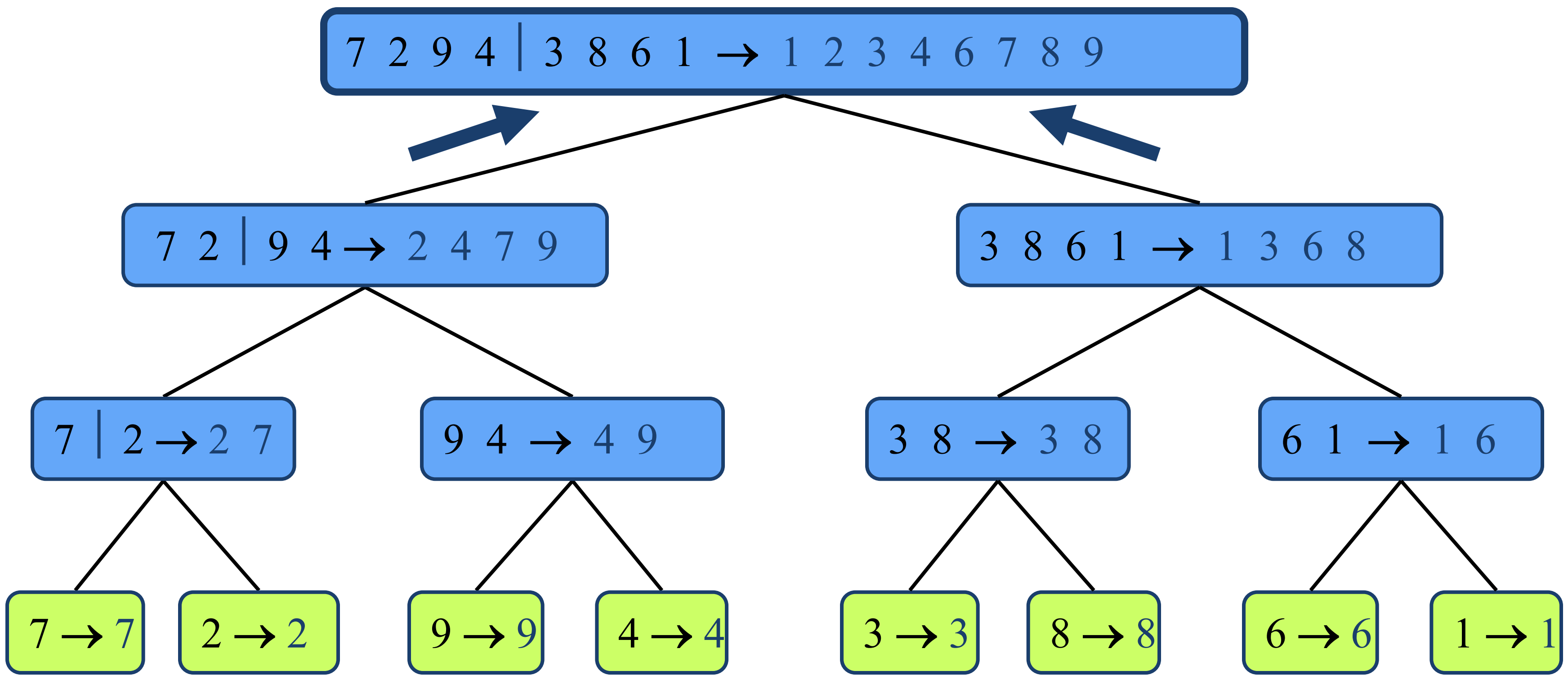


Execution Example (cont.)



Execution Example (cont.)

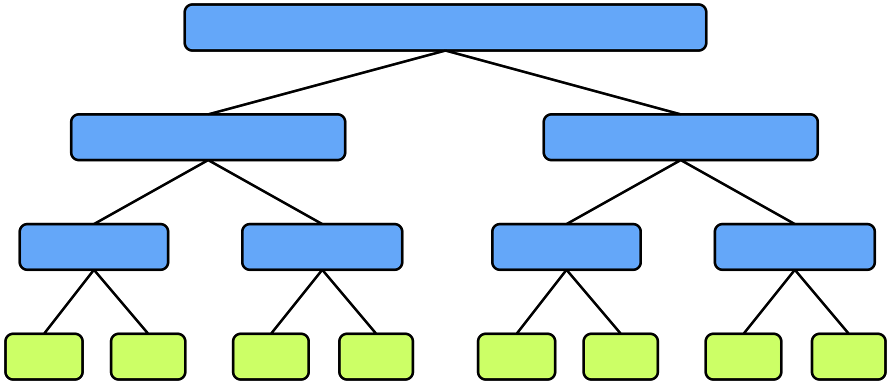
- Merge



Analysis of Merge-Sort

- The height h of the merge-sort tree is $O(\log n)$
  - at each recursive call we divide the sequence in half
- The overall amount or work done at the nodes of depth i is $O(n)$
  - we partition and merge 2^i sequences of size $n/2^i$
  - we make 2^{i+1} recursive calls
- Thus, the total running time of merge-sort is $O(n \log n)$

depth	#seqs	size
0	1	n
1	2	$n/2$
i	2^i	$n/2^i$
…	…	…



Comparing sorting algorithms

Consider the following when evaluating a sorting algorithm:

- Time complexity
- Space complexity
  - An in-place algorithm requires only $n + O(1)$ space, using the already given space for the n elements and $O(1)$ additional space
- Stability
  - A sorting algorithm is stable if it preserves the original relative ordering of elements with equal value
  - Ex: Unsorted sequence (**B**, **b**, a, c). Suppose B = b and a < b < c.
    - Stable sorted: (a, **B, b**, c)
    - Unstable sorted: (a, **b, B**, c)
  - Necessary if we want to sort repeatedly by different attributes (i.e., sort by first name, then sort again by last name)

Summary of Sorting Algorithms

Algorithm	Time	Notes
selection-sort	$O(n^2)$	◆ in-place ◆ not stable ◆ for small data sets (< 1K)
insertion-sort	$O(n^2)$	◆ in-place ◆ stable ◆ for small data sets (< 1K)
heap-sort	$O(n \log n)$	◆ in-place ◆ not stable ◆ for large data sets (1K — 1M)
merge-sort	$O(n \log n)$	◆ not in-place ◆ stable ◆ sequential data access ◆ for huge data sets (> 1M)

Other

- You are given a query point $\boldsymbol{p}$ and a set $\boldsymbol{S}$ of $\boldsymbol{n}$ other points in two dimensional space. Find $\boldsymbol{k}$ points out of the $\boldsymbol{n}$ points which are nearest to $\boldsymbol{p}$.