Greedy Method

Outline / Reading

- Greedy Method as a fundamental algorithm design technique

- Application to problems of:
  - Making change
  - Fractional Knapsack Problem (Ch. 5.1.1)
  - Task Scheduling (Ch. 5.1.2)
  - Minimum Spanning Trees (Ch. 7.3) [future lecture]

Greedy Method Technique

- The greedy method is a general algorithm design paradigm, built on the following elements:
  - configurations: different choices, collections, or values to find
  - objective function: a score assigned to configurations, which we want to either maximize or minimize
- Idea: make a greedy choice (locally optimal) in hopes it will eventually lead to a globally optimal solution.
- It works best when applied to problems with the greedy-choice property
  - a globally-optimal solution can always be found by a series of local improvements from a starting configuration.

Making Change

- Problem: A dollar amount to reach and a collection of coin amounts to use to get there.
  - configuration: A dollar amount yet to return to a customer plus the coins already returned
  - objective function: Minimize number of coins returned.
- Greedy solution: Always return the largest coin you can.

- Ex. 1: Coins are valued \$.32, \$.08, \$.01
  - Has the greedy-choice property, since no amount over \$.32 can be made with a minimum number of coins by omitting a \$.32 coin (similarly for amounts over \$.08, but under \$.32).
- Ex. 2: Coins are valued \$.30, \$.20, \$.05, \$.01
  - Does not have greedy-choice property, since \$.40 is best made with two \$.20's, but the greedy solution will pick three coins (which ones?)

Fractional Knapsack Problem

- Given: A set S of n items, with each item i having
  - b_i - a positive benefit
  - w_i - a positive weight
- Goal: Choose items with maximum total benefit but with weight at most W.

If we are allowed to take fractional amounts, then this is called the fractional knapsack problem.

- In this case, we let x_i denote the amount we take of item i
- objective: maximize

$$\sum_{i \in S} b_i (x_i / w_i)$$

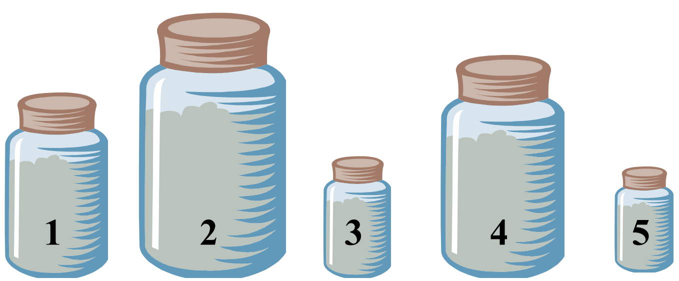
- constraint:

$$\sum_{i \in S} x_i \leq W$$

Example

- Given: A set S of n items, with each item i having
  - b_i - a positive benefit
  - w_i - a positive weight
- Goal: Choose items with maximum total benefit but with weight at most W.

Items:



Weight:	4 ml	8 ml	2 ml	6 ml	1 ml
Benefit:	$12	$32	$40	$30	$50
Value: ($ per ml)	3	4	20	5	50

“knapsack”

10 ml

Solution:
- 1 ml of item 5
- 2 ml of item 3
- 6 ml of item 4
- 1 ml of item 2

Fractional Knapsack Algorithm

Greedy choice: Keep taking item with highest **value** (benefit to weight ratio)

- Since $\sum_{i \in S} b_i (x_i / w_i) = \sum_{i \in S} (b_i / w_i) x_i$
- Run time: O(n log n). Why?

Correctness:

Suppose there is a optimal solution S* better than our greedy solution S.

- There is an item i in S with higher value than a chosen item j from S*, i.e., $v_i > v_j$ but $x_i < w_i$ and $x_j > 0$.
- If we substitute some i with j, we get a better solution in S*, a contradiction
  - How much of i: min$\{w_i - x_i, x_j\}$
- Thus, there is no better solution than the greedy one

Algorithm ***fractionalKnapsack***(***S***, ***W***)

Input: set S of items w/ benefit b_i and weight w_i; max. weight W

Output: amount x_i of each item i to maximize benefit with weight at most W

for ***each item i in S***

$x_i \leftarrow 0$

$v_i \leftarrow b_i / w_i$ {value}

$w \leftarrow 0$ {total weight}

while $w < W$

remove item i with highest v_i

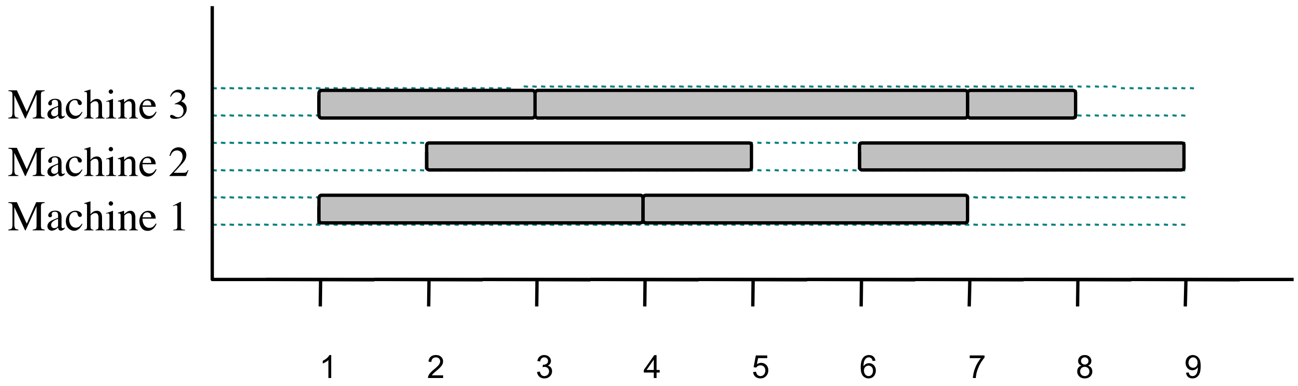
$x_i \leftarrow \min\{w_i, W - w\}$

$w \leftarrow w + x_i$

Task Scheduling

- Given: a set T of n tasks, each having:
  - A start time, s_i
  - A finish time, f_i (where $s_i < f_i$)
- Goal: Perform all the tasks using a minimum number of "machines."

Tasks: [3,7] [1,4] [1,3] [4,7] [6,9] [7,8] [2,5]



Task Scheduling Algorithm

Greedy choice: consider tasks by their start time and use as few machines as possible with this order.

- Run time: O(n log n). Why?

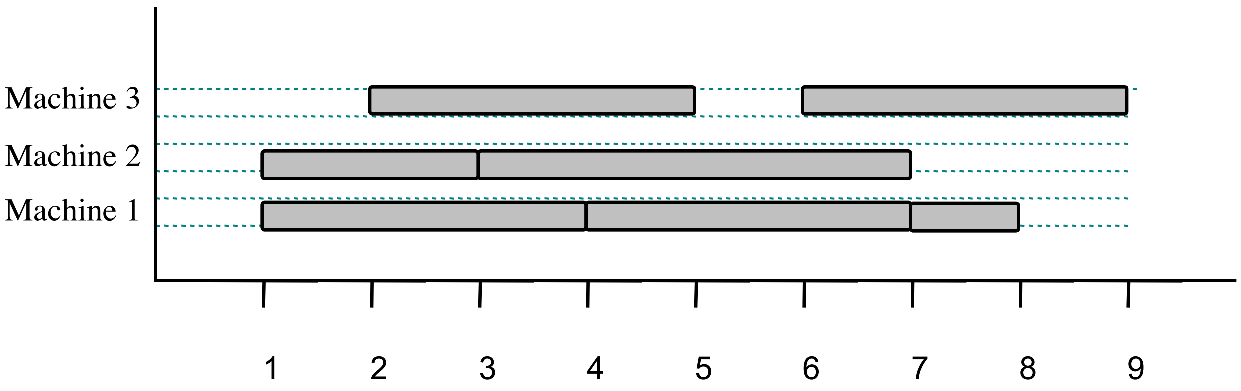
Correctness:
Suppose there is a better schedule.

- We can use *k-1* machines
- The algorithm uses *k*
- Let *i* be first task scheduled on machine *k*
- Task *i* must conflict with *k-1* other tasks
- But that means there is no non-conflicting schedule using *k-1* machines

```
Algorithm taskSchedule(T)
    Input: set T of tasks w/ start time s_i
    and finish time f_i
    Output: non-conflicting schedule
    with minimum number of machines
    m ← 0                    {no. of machines}
    while T is not empty
        remove task i w/ smallest s_i
        if there's a machine j for i then
            schedule i on machine j
        else
            m ← m + 1
            schedule i on machine m
```

Example

- Given: a set T of n tasks, each having:
  - A start time, s_i
  - A finish time, f_i (where $s_i < f_i$)
  - [1,4], [1,3], [2,5], [3,7], [4,7], [6,9], [7,8] (ordered by start)
- Goal: Perform all tasks on min. number of machines



Other

- The university has *n* classes it needs to schedule, using the minimum number of rooms possible.
  - Each class has a start/end time.
  - Each class should have at least 15 minutes between when one class ends in that room to when another class begins in the same room.