

Dynamic Programming

Outline and Reading

- Matrix Chain-Product (5.3.1)
- Dynamic Programming: The General Technique (5.3.2)
- 0-1 Knapsack Problem (5.3.3)

Matrix Chain Product

Dynamic Programming is a general algorithm design paradigm.

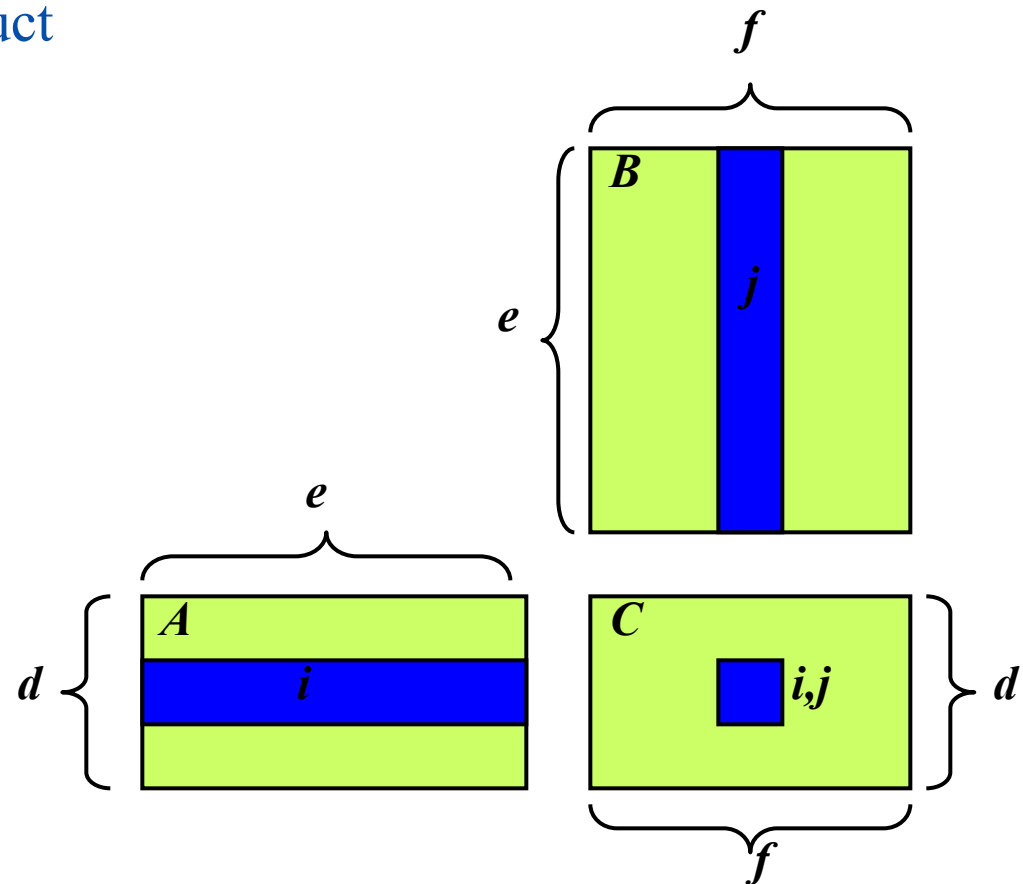
- Rather than give the general structure, we first give a motivating example: **Matrix Chain-Product**

Review: Matrix Multiplication

- $C = A * B$
- A is $d \times e$ and B is $e \times f$

$$C[i, j] = \sum_{k=0}^{e-1} A[i, k] * B[k, j]$$

- $O(d \cdot e \cdot f)$ time



Matrix Chain Product

Matrix Chain-Product:

- Compute $A = A_0 * A_1 * \dots * A_{n-1}$
- A_i is $d_i \times d_{i+1}$
- **Problem:** How to parenthesize in such a way that **minimizes the total number of scalar multiplications?**

Example:

- B is 3×100
- C is 100×5
- D is 5×5
- $(B * C) * D$ takes $1500 + 75 = 1575$ ops
- $B * (C * D)$ takes $1500 + 2500 = 4000$ ops

Another Approach: Greedy (v1)

Idea: Repeatedly select the product that uses (up) the most operations.

Counter-example:

- A is 10×5
- B is 5×10
- C is 10×5
- D is 5×10

This greedy approach gives $(A * B) * (C * D)$

- takes $500 + 1000 + 500 = 2000$ ops

A better solution: $A * ((B * C) * D)$

- takes $500 + 250 + 250 = 1000$ ops

Another Approach: Greedy (v2)

Idea: Repeatedly select the product that uses the fewest operations.

Counter-example:

- A is 101×11
- B is 11×9
- C is 9×100
- D is 100×99

This greedy approach gives $A * ((B * C) * D)$

- takes $109989 + 9900 + 108900 = 228789$ ops

A better solution is $(A * B) * (C * D)$

- takes $9999 + 89991 + 89100 = 189090$ ops

The greedy approach is not giving us the optimal value.

“Recursive” Approach

Define **subproblems**:

- Find the best parenthesization of $A_i * A_{i+1} * \dots * A_j$.
- Let $N_{i,j}$ denote the number of operations done by this subproblem.
- The optimal solution for the whole problem is $N_{0,n-1}$.

Subproblem optimality: The optimal solution can be defined in terms of optimal subproblems

- There has to be a final multiplication (root of the expression tree) for the optimal solution.
- Say, the final multiply is at index i : $(A_0 * \dots * A_i) * (A_{i+1} * \dots * A_{n-1})$.
- Then the optimal solution $N_{0,n-1}$ is the sum of two optimal subproblems, $N_{0,i}$ and $N_{i+1,n-1}$ plus the time for the last multiply.
- If the global optimum did not have these optimal subproblems, we could define an even better “optimal” solution.

Characterizing Equation

- The global optimal has to be defined in terms of optimal subproblems, depending on where the final multiply is at.
- Consider all possible places for that final multiply:
 - Recall that A_i is a $d_i \times d_{i+1}$ dimensional matrix.
 - So, a characterizing equation for $N_{i,j}$ is the following:

$$N_{i,j} = \min_{i \leq k < j} \{ N_{i,k} + N_{k+1,j} + d_i d_{k+1} d_{j+1} \}$$

- Note that subproblems are not independent – meaning **subproblems overlap**.

Dynamic Programming Algorithm Visualization

The bottom-up construction fills in the N array by diagonals

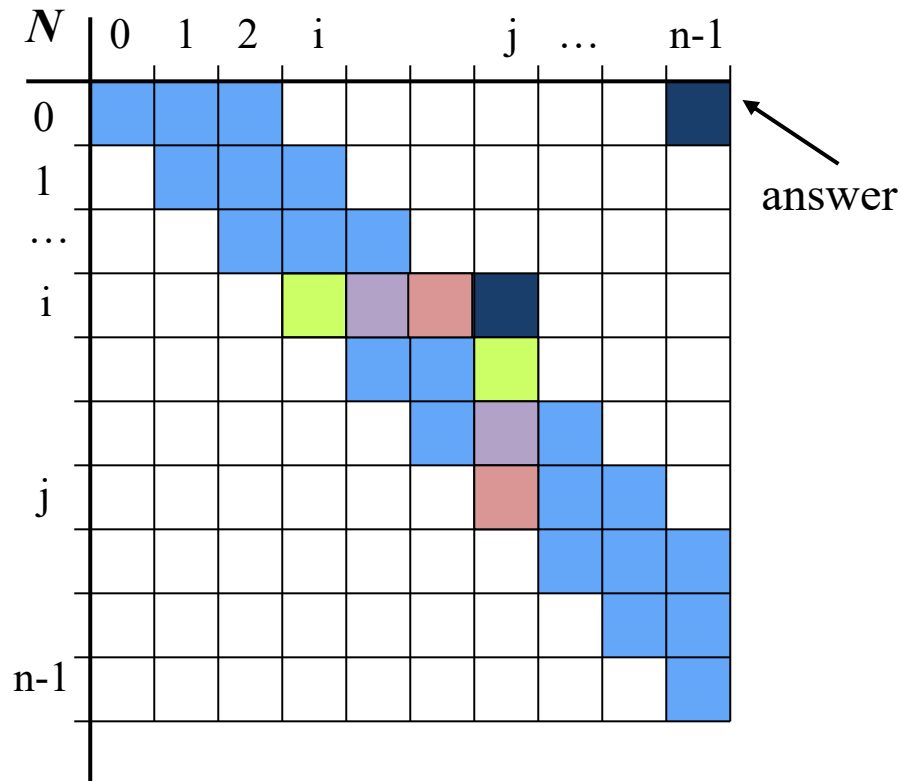
$$N_{i,j} = \min_{i \leq k < j} \{N_{i,k} + N_{k+1,j} + d_i d_{k+1} d_{j+1}\}$$

$N_{i,j}$ gets values from previous entries in i -th row and j -th column

Filling in each entry in the N table takes $O(n)$ time.

- Total run time: $O(n^3)$

Getting actual parenthesization can be done by remembering “k” for each N entry in a separate table



Dynamic Programming Algorithm

Since subproblems overlap, we don't use recursion.

Instead, we construct optimal subproblems "bottom-up."

$N_{i,i}$'s are easy, so start with them

Then do problems of "length" 2,3,... subproblems, and so on.

Running time: $O(n^3)$

Algorithm *matrixChain*(S):

Input: sequence S of n matrices to be multiplied

Output: number of operations in an optimal parenthesization of S

for $i \leftarrow 0$ **to** $n - 1$ **do**

$N_{i,i} \leftarrow 0$

for $length \leftarrow 1$ **to** $n - 1$ **do**

{ $length = j - i$ is the length of the chain }

for $i \leftarrow 0$ **to** $n - 1 - length$ **do**

$j \leftarrow i + length$

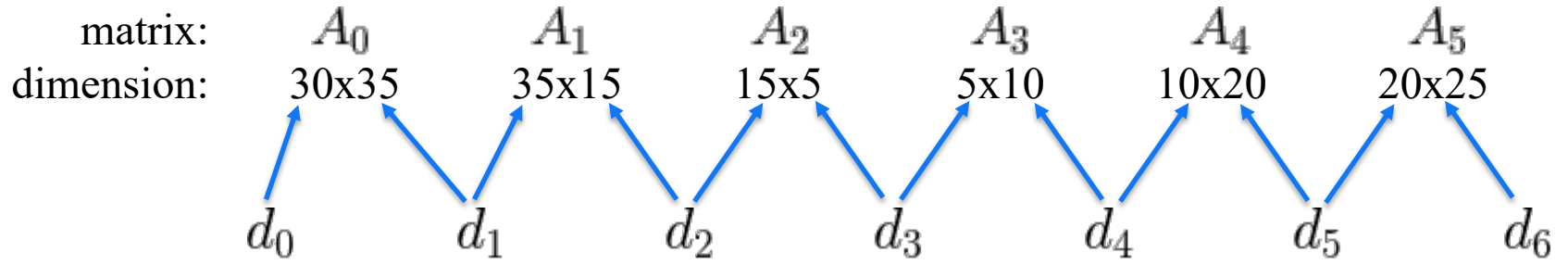
$N_{i,j} \leftarrow +\infty$

for $k \leftarrow i$ **to** $j - 1$ **do**

$N_{i,j} \leftarrow \min \{ N_{i,j}, N_{i,k} + N_{k+1,j} + d_i d_{k+1} d_{j+1} \}$

record k that produces minimum $N_{i,j}$

return $N_{0,n-1}$



matrix: A_0 A_1 A_2 A_3 A_4 A_5
dimension: 30x35 35x15 15x5 5x10 10x20 20x25
 d_0 d_1 d_2 d_3 d_4 d_5 d_6

end j

N	0	1	2	3	4	5
0	0					
1		0				
2			0			
3				0		
4					0	
5						0

start i

number of scalar operations required to multiply
 $A_i \cdot A_{i+1} \cdot A_{i+2} \cdot \dots \cdot A_{j-1} \cdot A_j$

k	0	1	2	3	4	5
0						
1						
2						
3						
4						
5						

matrix index where final multiplication occurred to obtain optimal solution given in $N[i][j]$

matrix: A_0 A_1 A_2 A_3 A_4 A_5
 dimension: 30×35 35×15 15×5 5×10 10×20 20×25
 d_0 d_1 d_2 d_3 d_4 d_5 d_6

end j

N	0	1	2	3	4	5
0	0	15750				
1		0				
2			0			
3				0		
4					0	
5						0

start i

k	0	1	2	3	4	5
0		0				
1						
2						
3						
4						
5						

number of scalar operations required to multiply

$$A_i \cdot A_{i+1} \cdot A_{i+2} \cdot \dots \cdot A_{j-1} \cdot A_j$$

$$A_0 \cdot A_1 \quad N[0][1] = 0 + 0 + 30 \cdot 35 \cdot 15 = 15750$$

matrix index where final multiplication occurred to obtain optimal solution given in $N[i][j]$

matrix: A_0 A_1 A_2 A_3 A_4 A_5
dimension: 30x35 35x15 15x5 5x10 10x20 20x25
 d_0 d_1 d_2 d_3 d_4 d_5 d_6

end j

N	0	1	2	3	4	5
0	0	15750				
1		0	2625			
2			0			
3				0		
4					0	
5						0

start i

k	0	1	2	3	4	5
0		0				
1			1			
2						
3						
4						
5						

number of scalar operations required to multiply

$$A_i \cdot A_{i+1} \cdot A_{i+2} \cdot \dots \cdot A_{j-1} \cdot A_j$$

$$A_0 \cdot A_1 \quad N[0][1] = 0 + 0 + 30 \cdot 35 \cdot 15 = 15750$$

$$A_1 \cdot A_2 \quad N[1][2] = 0 + 0 + 35 \cdot 15 \cdot 5 = 2626$$

matrix index where final multiplication occurred to obtain optimal solution given in $N[i][j]$

matrix: A_0 A_1 A_2 A_3 A_4 A_5
dimension: 30x35 35x15 15x5 5x10 10x20 20x25
 d_0 d_1 d_2 d_3 d_4 d_5 d_6

end j

N	0	1	2	3	4	5
0	0	15750				
1		0	2625			
2			0	750		
3				0		
4					0	
5						0

start i

k	0	1	2	3	4	5
0		0				
1			1			
2				2		
3						
4						
5						

number of scalar operations required to multiply

$$A_i \cdot A_{i+1} \cdot A_{i+2} \cdot \dots \cdot A_{j-1} \cdot A_j$$

$$A_0 \cdot A_1 \quad N[0][1] = 0 + 0 + 30 \cdot 35 \cdot 15 = 15750$$

$$A_1 \cdot A_2 \quad N[1][2] = 0 + 0 + 35 \cdot 15 \cdot 5 = 2626$$

$$A_2 \cdot A_3 \quad N[2][3] = 0 + 0 + 15 \cdot 5 \cdot 10 = 750$$

matrix index where final multiplication occurred to obtain optimal solution given in $N[i][j]$

matrix: A_0 A_1 A_2 A_3 A_4 A_5
dimension: 30x35 35x15 15x5 5x10 10x20 20x25
 d_0 d_1 d_2 d_3 d_4 d_5 d_6

end j

N	0	1	2	3	4	5
0	0	15750				
1		0	2625			
2			0	750		
3				0	1000	
4					0	
5						0

start i

k	0	1	2	3	4	5
0		0				
1			1			
2				2		
3					3	
4						
5						

number of scalar operations required to multiply

$$A_i \cdot A_{i+1} \cdot A_{i+2} \cdot \dots \cdot A_{j-1} \cdot A_j$$

- $A_0 \cdot A_1$ $N[0][1] = 0 + 0 + 30 \cdot 35 \cdot 15 = 15750$
- $A_1 \cdot A_2$ $N[1][2] = 0 + 0 + 35 \cdot 15 \cdot 5 = 2626$
- $A_2 \cdot A_3$ $N[2][3] = 0 + 0 + 15 \cdot 5 \cdot 10 = 750$
- $A_3 \cdot A_4$ $N[3][4] = 0 + 0 + 5 \cdot 10 \cdot 20 = 1000$

matrix index where final multiplication occurred to obtain optimal solution given in $N[i][j]$

matrix: A_0 A_1 A_2 A_3 A_4 A_5
dimension: 30x35 35x15 15x5 5x10 10x20 20x25
 d_0 d_1 d_2 d_3 d_4 d_5 d_6

end j

N	0	1	2	3	4	5
0	0	15750				
1		0	2625			
2			0	750		
3				0	1000	
4					0	5000
5						0

start i

k	0	1	2	3	4	5
0		0				
1			1			
2				2		
3					3	
4						4
5						

number of scalar operations required to multiply

$$A_i \cdot A_{i+1} \cdot A_{i+2} \cdot \dots \cdot A_{j-1} \cdot A_j$$

$$A_0 \cdot A_1 \quad N[0][1] = 0 + 0 + 30 \cdot 35 \cdot 15 = 15750$$

$$A_1 \cdot A_2 \quad N[1][2] = 0 + 0 + 35 \cdot 15 \cdot 5 = 2626$$

$$A_2 \cdot A_3 \quad N[2][3] = 0 + 0 + 15 \cdot 5 \cdot 10 = 750$$

$$A_3 \cdot A_4 \quad N[3][4] = 0 + 0 + 5 \cdot 10 \cdot 20 = 1000$$

$$A_4 \cdot A_5 \quad N[4][5] = 0 + 0 + 10 \cdot 20 \cdot 25 = 5000$$

matrix index where final multiplication occurred to obtain optimal solution given in $N[i][j]$

matrix: A_0 A_1 A_2 A_3 A_4 A_5
 dimension: 30×35 35×15 15×5 5×10 10×20 20×25
 d_0 d_1 d_2 d_3 d_4 d_5 d_6

end j

N	0	1	2	3	4	5
0	0	15750	7875			
1		0	2625			
2			0	750		
3				0	1000	
4					0	5000
5						0

start i

k	0	1	2	3	4	5
0		0	0			
1			1			
2				2		
3					3	
4						4
5						

$$(A_0) \cdot (A_1 \cdot A_2) = 0 + 2625 + 30 \cdot 35 \cdot 5 = 7875$$

$$(A_0 \cdot A_1) \cdot (A_2) = 15750 + 0 + 30 \cdot 15 \cdot 5 = 18000$$

matrix: A_0 A_1 A_2 A_3 A_4 A_5
dimension: 30×35 35×15 15×5 5×10 10×20 20×25
 d_0 d_1 d_2 d_3 d_4 d_5 d_6

end j

N	0	1	2	3	4	5
0	0	15750	7875			
1		0	2625	4375		
2			0	750		
3				0	1000	
4					0	5000
5						0

start i

k	0	1	2	3	4	5
0		0	0			
1			1	2		
2				2		
3					3	
4						4
5						

$$(A_1) \cdot (A_2 \cdot A_3) = 0 + 750 + 35 \cdot 15 \cdot 10 = 6000$$

$$(A_1 \cdot A_2) \cdot (A_3) = 2625 + 0 + 35 \cdot 5 \cdot 10 = 4375$$

matrix: A_0 A_1 A_2 A_3 A_4 A_5
dimension: 30×35 35×15 15×5 5×10 10×20 20×25
 d_0 d_1 d_2 d_3 d_4 d_5 d_6

end j

N	0	1	2	3	4	5
0	0	15750	7875			
1		0	2625	4375		
2			0	750	2500	
3				0	1000	
4					0	5000
5						0

start i

k	0	1	2	3	4	5
0		0	0			
1			1	2		
2				2	2	
3					3	
4						4
5						

$$(A_2) \cdot (A_3 \cdot A_4) = 0 + 1000 + 15 \cdot 5 \cdot 20 = 2500$$

$$(A_2 \cdot A_3) \cdot (A_4) = 750 + 0 + 15 \cdot 10 \cdot 20 = 3750$$

matrix: A_0 A_1 A_2 A_3 A_4 A_5
 dimension: 30×35 35×15 15×5 5×10 10×20 20×25
 d_0 d_1 d_2 d_3 d_4 d_5 d_6

end j

start i

N	0	1	2	3	4	5
0	0	15750	7875			
1		0	2625	4375		
2			0	750	2500	
3				0	1000	3500
4					0	5000
5						0

k	0	1	2	3	4	5
0		0	0			
1			1	2		
2				2	2	
3					3	4
4						4
5						

$$(A_3) \cdot (A_4 \cdot A_5) = 0 + 5000 + 5 \cdot 10 \cdot 25 = 6250$$

$$(A_3 \cdot A_4) \cdot (A_5) = 1000 + 0 + 5 \cdot 20 \cdot 25 = 3500$$

matrix: A_0 A_1 A_2 A_3 A_4 A_5
 dimension: 30×35 35×15 15×5 5×10 10×20 20×25
 d_0 d_1 d_2 d_3 d_4 d_5 d_6

end j

N	0	1	2	3	4	5
0	0	15750	7875	9375		
1		0	2625	4375		
2			0	750	2500	
3				0	1000	3500
4					0	5000
5						0

start i

k	0	1	2	3	4	5
0		0	0	2		
1			1	2		
2				2	2	
3					3	4
4						4
5						

$$(A_0) \cdot (A_1 \cdot A_2 \cdot A_3) = 0 + 4375 + 30 \cdot 35 \cdot 10 = 14875$$

$$(A_0 \cdot A_1) \cdot (A_2 \cdot A_3) = 15750 + 750 + 30 \cdot 15 \cdot 10 = 21000$$

$$(A_0 \cdot A_1 \cdot A_2) \cdot (A_3) = 7875 + 0 + 30 \cdot 5 \cdot 10 = 9375$$

matrix: A_0 A_1 A_2 A_3 A_4 A_5
dimension: 30×35 35×15 15×5 5×10 10×20 20×25
 d_0 d_1 d_2 d_3 d_4 d_5 d_6

end j

N	0	1	2	3	4	5
0	0	15750	7875	9375		
1		0	2625	4375	7125	
2			0	750	2500	
3				0	1000	3500
4					0	5000
5						0

start i

k	0	1	2	3	4	5
0		0	0	2		
1			1	2	2	
2				2	2	
3					3	4
4						4
5						

$$(A_1) \cdot (A_2 \cdot A_3 \cdot A_4) = 0 + 2500 + 35 \cdot 15 \cdot 20 = 13000$$

$$(A_1 \cdot A_2) \cdot (A_3 \cdot A_4) = 2625 + 1000 + 35 \cdot 5 \cdot 20 = 7125$$

$$(A_1 \cdot A_2 \cdot A_3) \cdot (A_4) = 4375 + 0 + 35 \cdot 10 \cdot 20 = 11375$$

matrix: A_0 A_1 A_2 A_3 A_4 A_5
 dimension: 30×35 35×15 15×5 5×10 10×20 20×25
 d_0 d_1 d_2 d_3 d_4 d_5 d_6

end j

start i

N	0	1	2	3	4	5
0	0	15750	7875	9375		
1		0	2625	4375	7125	
2			0	750	2500	5375
3				0	1000	3500
4					0	5000
5						0

k	0	1	2	3	4	5
0		0	0	2		
1			1	2	2	
2				2	2	2
3					3	4
4						4
5						

$$(A_2) \cdot (A_3 \cdot A_4 \cdot A_5) = 0 + 3500 + 15 \cdot 5 \cdot 25 = 5375$$

$$(A_2 \cdot A_3) \cdot (A_4 \cdot A_5) = 750 + 5000 + 15 \cdot 10 \cdot 25 = 9500$$

$$(A_2 \cdot A_3 \cdot A_4) \cdot (A_5) = 2500 + 0 + 15 \cdot 20 \cdot 25 = 10000$$

matrix: A_0 A_1 A_2 A_3 A_4 A_5
dimension: 30×35 35×15 15×5 5×10 10×20 20×25
 d_0 d_1 d_2 d_3 d_4 d_5 d_6

end j

N	0	1	2	3	4	5
0	0	15750	7875	9375	11875	
1		0	2625	4375	7125	
2			0	750	2500	5375
3				0	1000	3500
4					0	5000
5						0

start i

k	0	1	2	3	4	5
0		0	0	2	2	
1			1	2	2	
2				2	2	2
3					3	4
4						4
5						

$$(A_0) \cdot (A_1 \cdot A_2 \cdot A_3 \cdot A_4) = 0 + 7125 + 30 \cdot 35 \cdot 20 = 28125$$

$$(A_0 \cdot A_1) \cdot (A_2 \cdot A_3 \cdot A_4) = 15750 + 2500 + 30 \cdot 15 \cdot 20 = 27250$$

$$(A_0 \cdot A_1 \cdot A_2) \cdot (A_3 \cdot A_4) = 7875 + 1000 + 30 \cdot 5 \cdot 20 = 11875$$

$$(A_0 \cdot A_1 \cdot A_2 \cdot A_3) \cdot (A_4) = 9375 + 0 + 30 \cdot 10 \cdot 20 = 15375$$

matrix: A_0 A_1 A_2 A_3 A_4 A_5
dimension: 30x35 35x15 15x5 5x10 10x20 20x25
 d_0 d_1 d_2 d_3 d_4 d_5 d_6

end j

N	0	1	2	3	4	5
0	0	15750	7875	9375	11875	
1		0	2625	4375	7125	10500
2			0	750	2500	5375
3				0	1000	3500
4					0	5000
5						0

start i

k	0	1	2	3	4	5
0		0	0	2	2	
1			1	2	2	2
2				2	2	2
3					3	4
4						4
5						

$$\begin{aligned}
 (A_1) \cdot (A_2 \cdot A_3 \cdot A_4 \cdot A_5) &= 0 + 5375 + 35 \cdot 15 \cdot 25 = 18500 \\
 (A_1 \cdot A_2) \cdot (A_3 \cdot A_4 \cdot A_5) &= 2625 + 3500 + 35 \cdot 5 \cdot 25 = 10500 \\
 (A_1 \cdot A_2 \cdot A_3) \cdot (A_4 \cdot A_5) &= 4375 + 5000 + 35 \cdot 10 \cdot 25 = 18125 \\
 (A_1 \cdot A_2 \cdot A_3 \cdot A_4) \cdot (A_5) &= 7125 + 0 + 35 \cdot 20 \cdot 25 = 24625
 \end{aligned}$$

matrix: A_0 A_1 A_2 A_3 A_4 A_5
dimension: 30x35 35x15 15x5 5x10 10x20 20x25
 d_0 d_1 d_2 d_3 d_4 d_5 d_6

end j

N	0	1	2	3	4	5
0	0	15750	7875	9375	11875	15125
1		0	2625	4375	7125	10500
2			0	750	2500	5375
3				0	1000	3500
4					0	5000
5						0

start i

k	0	1	2	3	4	5
0		0	0	2	2	2
1			1	2	2	2
2				2	2	2
3					3	4
4						4
5						

$$(A_0) \cdot (A_1 \cdot A_2 \cdot A_3 \cdot A_4 \cdot A_5) = 0 + 10500 + 30 \cdot 35 \cdot 25 = 36750$$

$$(A_0 \cdot A_1) \cdot (A_2 \cdot A_3 \cdot A_4 \cdot A_5) = 15750 + 5375 + 30 \cdot 15 \cdot 25 = 32375$$

$$(A_0 \cdot A_1 \cdot A_2) \cdot (A_3 \cdot A_4 \cdot A_5) = 7875 + 3500 + 30 \cdot 5 \cdot 25 = 15125$$

$$(A_0 \cdot A_1 \cdot A_2 \cdot A_3) \cdot (A_4 \cdot A_5) = 9375 + 5000 + 30 \cdot 10 \cdot 25 = 21875$$

$$(A_0 \cdot A_1 \cdot A_2 \cdot A_3 \cdot A_4) \cdot (A_5) = 11875 + 0 + 30 \cdot 20 \cdot 25 = 26875$$

matrix: A_0 A_1 A_2 A_3 A_4 A_5
 dimension: 30×35 35×15 15×5 5×10 10×20 20×25
 d_0 d_1 d_2 d_3 d_4 d_5 d_6

end j

start i

N	0	1	2	3	4	5
0	0	15750	7875	9375	11875	15125
1		0	2625	4375	7125	10500
2			0	750	2500	5375
3				0	1000	3500
4					0	5000
5						0

k	0	1	2	3	4	5
0		0	0	2	2	2
1			1	2	2	2
2				2	2	2
3					3	4
4						4
5						

optimal order in which the following matrices should be multiplied:

$$[(A_0) \cdot (A_1 \cdot A_2)] \cdot [(A_3 \cdot A_4) \cdot (A_5)]$$

General Dynamic Programming Technique

Applies to an optimization problem that at first seems to require a lot of time (possibly exponential), provided we have:

- **Simple subproblems:** the subproblems can be defined in terms of a few variables, such as j , k , l , m , and so on.
- **Subproblem overlap:** the subproblems are not independent, but instead they overlap (hence, should be constructed bottom-up).
- **Subproblem optimality:** the global optimum value can be defined in terms of optimal subproblems

0/1 Knapsack Problem



Given: A set S of n items, with each item i having

- w_i - a positive weight
- b_i - a positive benefit

Goal: Choose items with maximum total benefit but with weight at most W .

If we are **not** allowed to take fractional amounts, then this is the **0/1 knapsack problem**.

- In this case, we let T denote the set of items we take
- Objective: maximize
- Constraint:

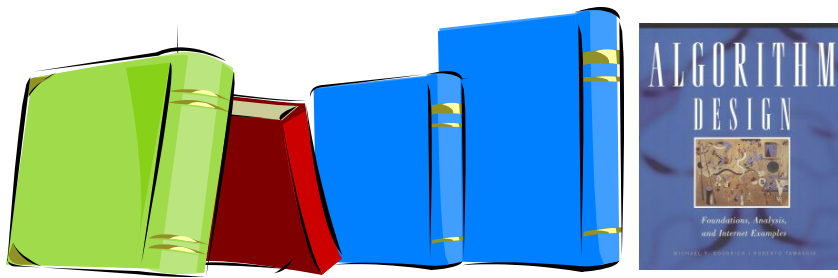
$$\sum_{i \in T} b_i$$

$$\sum_{i \in T} w_i \leq W$$

Example

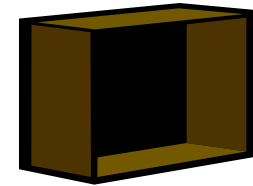
- Given: A set S of n items, with each item i having
 - b_i - a positive “benefit”
 - w_i - a positive “weight”
- Goal: Choose items with maximum total benefit but with weight at most W .

Items:



	1	2	3	4	5
Weight:	4 in	2 in	2 in	6 in	2 in
Benefit:	\$20	\$3	\$6	\$25	\$80

“knapsack”



box of width 9 in

Solution:

- item 5 (\$80, 2 in)
- item 3 (\$6, 2in)
- item 1 (\$20, 4in)

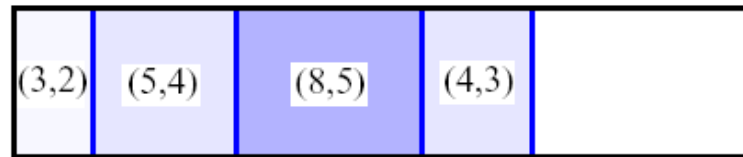
A 0/1 Knapsack Algorithm: First Attempt



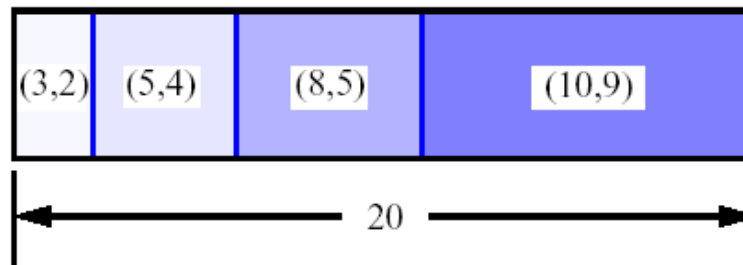
S_k : Set of items numbered 1 to k .

- Idea: Define $B[k] =$ best selection from S_k .
- Problem: does **not** have subproblem optimality.
 - Consider set $S = \{(3,2), (5,4), (8,5), (4,3), (10,9)\}$ of (benefit, weight) pairs and total weight $W = 20$

Best for S_4 :



Best for S_5 :



A 0/1 Knapsack Algorithm: Second Attempt



S_k : Set of items numbered 1 to k .

- Idea: Define $B[k, w]$ to be the best selection from S_k with weight at most w
- Good news: this does have subproblem optimality.

$$B[k, w] = \begin{cases} B[k-1, w] & \text{if } w_k > w \\ \max \{B[k-1, w], B[k-1, w-w_k] + b_k\} & \text{else} \end{cases}$$

That is, the best subset of S_k with weight at most w is either

- the best subset of S_{k-1} with weight at most w or
- the best subset of S_{k-1} with weight at most $w-w_k$ plus item k

0/1 Knapsack Algorithm



$$B[k, w] = \begin{cases} B[k-1, w] & \text{if } w_k > w \\ \max \{B[k-1, w], B[k-1, w-w_k] + b_k\} & \text{else} \end{cases}$$

- Recall the definition of $B[k, w]$
- Since $B[k, w]$ is defined in terms of $B[k-1, *]$, we can use two arrays of instead of a matrix
- Running time: $O(nW)$.
- Not a polynomial-time algorithm since W may be large
- This is a pseudo-polynomial time algorithm

Algorithm *01Knapsack*(S, W):

Input: set S of n items with benefit b_i and weight w_i ; maximum weight W

Output: benefit of best subset of S with weight at most W

let A and B be arrays of length $W + 1$

for $w \leftarrow 0$ to W do

$B[w] \leftarrow 0$

for $k \leftarrow 1$ to n do

 copy array B into array A

 for $w \leftarrow w_k$ to W do

 if $A[w-w_k] + b_k > A[w]$ then

$B[w] \leftarrow A[w-w_k] + b_k$

return $B[W]$