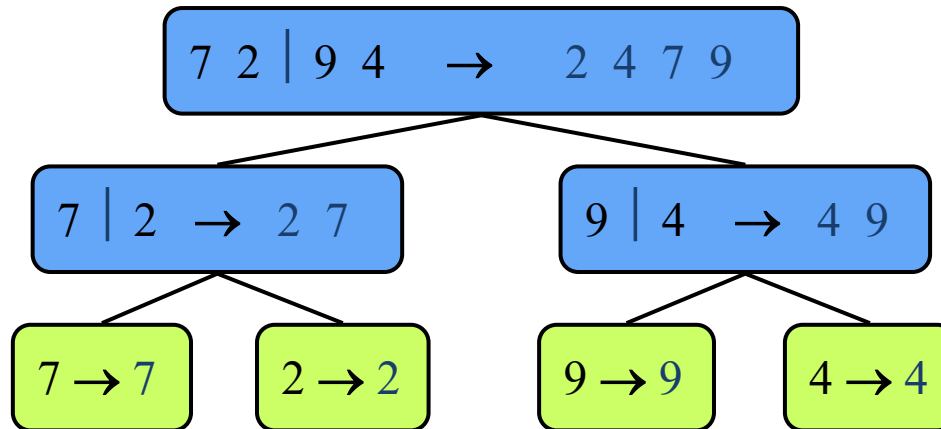


Divide and Conquer

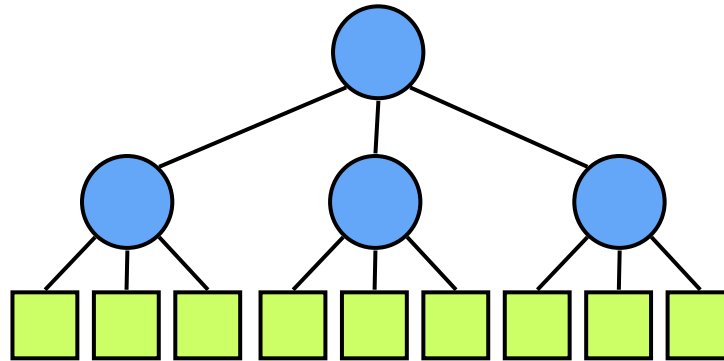


Outline / Reading

- Divide-and-conquer paradigm (5.2)
- Review Merge-sort (4.1.1)
- Recurrence Equations (5.2.1)
 - Recursion trees
 - Induction
 - Iterative substitution
 - Guess-and-test
 - The master method
- Integer Multiplication (5.2.2)

Divide-and-Conquer

- **Divide-and conquer** is a general algorithm design paradigm:
 - **Divide**: divide the input data in two or more disjoint subsets $\mathcal{S}_1, \mathcal{S}_2, \dots$
 - **Recur**: solve the subproblems recursively
 - **Conquer**: combine the solutions for $\mathcal{S}_1, \mathcal{S}_2, \dots$, into a solution for \mathcal{S}
- The base case for the recursion are subproblems of constant size
- Analysis can be done using **recurrence equations**



Merge Sort Review

Merge-sort on an input sequence S with n elements consists of three steps:

- **Divide:** partition S into two sequences S_1 and S_2 of about $n/2$ elements each
- **Recur:** recursively sort S_1 and S_2
- **Conquer:** merge S_1 and S_2 into a unique sorted sequence

Algorithm *mergeSort*(S, C)

Input sequence S with n elements, comparator C

Output sequence S sorted according to C

if $S.size() > 1$

$(S_1, S_2) \leftarrow partition(S, n/2)$

mergeSort(S_1, C)

mergeSort(S_2, C)

$S \leftarrow merge(S_1, S_2)$

Recurrence Equation Analysis

- The conquer step of merge-sort consists of merging two sorted sequences, each with $n/2$ elements and implemented by means of a doubly linked list, takes at most bn steps, for some constant b .
- Likewise, the basis case ($n < 2$) will take at most b steps.
- Therefore, if we let $T(n)$ denote the running time of merge-sort:

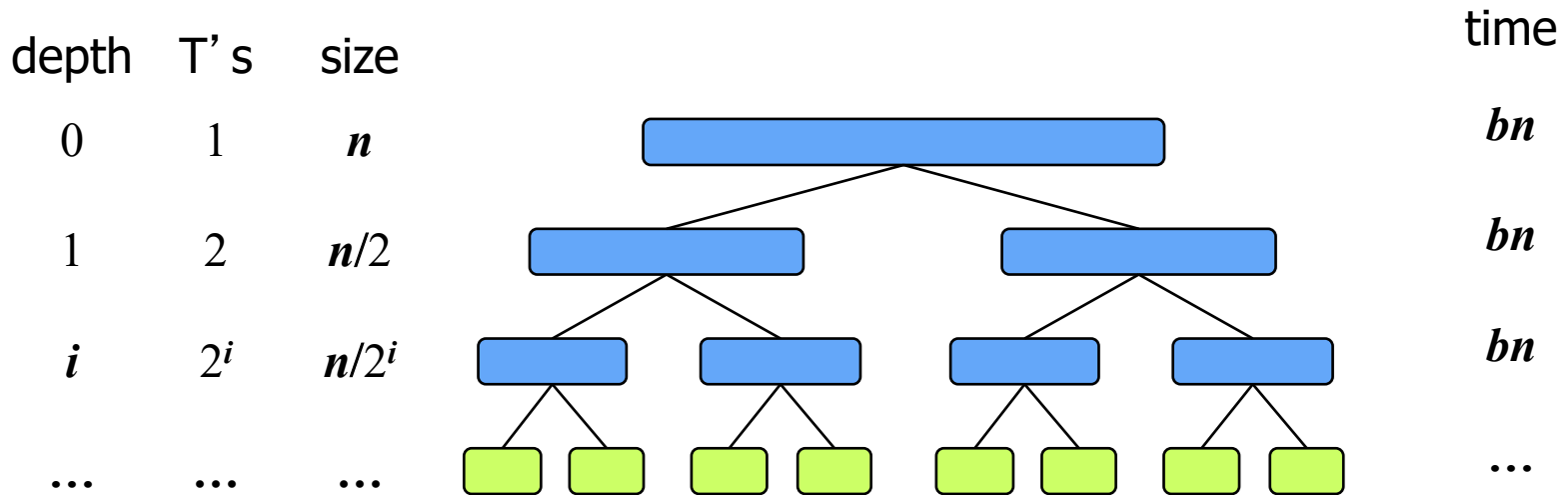
$$T(n) = \begin{cases} b & \text{if } n < 2 \\ 2T(n/2) + bn & \text{if } n \geq 2 \end{cases}$$

- We can analyze the running time of merge-sort by finding a **closed form solution** to the above equation.
 - That is, a solution that has $T(n)$ only on the left-hand side.

Recursion Tree

Draw the **recursion tree** for the recurrence relation and look for a pattern:

$$T(n) = \begin{cases} b & \text{if } n < 2 \\ 2T(n/2) + bn & \text{if } n \geq 2 \end{cases}$$



Total time = $bn + bn \log n$

(last level plus all previous levels)

Iterative Substitution

In the **iterative substitution**, or “plug-and-chug,” technique, we iteratively apply the recurrence equation to itself and see if we can find a pattern, then prove it is true by **induction**:

$$\begin{aligned}T(n) &= 2T(n/2) + bn \\&= 2(2T(n/2^2)) + b(n/2) + bn \\&= 2^2T(n/2^2) + 2bn \\&= 2^3T(n/2^3) + 3bn \\&= 2^4T(n/2^4) + 4bn \\&= \dots \\&= 2^i T(n/2^i) + ibn\end{aligned}$$

- Note that the base case, $T(n) = b$, case occurs when $2^i = n$. That is, $i = \log n$. So we have: $T(n) = bn + bn \log n$
- Once we prove this by induction, then $T(n)$ is $O(n \log n)$.

Guess-and-Test Method

In the **guess-and-test method**, we guess a closed form solution and then try to prove it is true by **induction**:

$$T(n) = \begin{cases} b & \text{if } n < 2 \\ 2T(n/2) + bn \log n & \text{if } n \geq 2 \end{cases}$$

- Guess #1: $T(n) \leq cn \log n$.

$$\begin{aligned} T(n) &= 2T(n/2) + bn \log n \\ &\leq 2(c(n/2) \log(n/2)) + bn \log n \\ &= cn(\log n - \log 2) + bn \log n \\ &= cn \log n - cn + bn \log n \end{aligned}$$

- **Wrong**: we cannot make this last line be less than $cn \log n$

Guess-and-Test Method (2)

Recall the recurrence equation:

$$T(n) = \begin{cases} b & \text{if } n < 2 \\ 2T(n/2) + bn \log n & \text{if } n \geq 2 \end{cases}$$

- Guess #2: $T(n) \leq cn \log^2 n$.

$$\begin{aligned} T(n) &= 2T(n/2) + bn \log n \\ &\leq 2(c(n/2) \log^2(n/2)) + bn \log n \\ &= cn(\log n - \log 2)^2 + bn \log n \\ &= cn \log^2 n - 2cn \log n + cn + bn \log n \\ &\leq cn \log^2 n \quad \text{if } c > b. \end{aligned}$$

- So, $T(n)$ is $O(n \log^2 n)$.

In general, to use this method, you need to have a good guess.

Master Method

Many divide-and-conquer recurrence equations have the form:

$$T(n) = \begin{cases} c & \text{if } n < d \\ aT(n/b) + f(n) & \text{if } n \geq d \end{cases}$$

The Master Theorem:

1. if $f(n)$ is $O(n^{\log_b a - \epsilon})$, then $T(n)$ is $\Theta(n^{\log_b a})$
2. if $f(n)$ is $\Theta(n^{\log_b a} \log^k n)$, then $T(n)$ is $\Theta(n^{\log_b a} \log^{k+1} n)$
3. if $f(n)$ is $\Omega(n^{\log_b a + \epsilon})$, then $T(n)$ is $\Theta(f(n))$,
provided $af(n/b) \leq \delta f(n)$ for some $\delta < 1$.

Master Method: Ex. 1

The form:
$$T(n) = \begin{cases} c & \text{if } n < d \\ aT(n/b) + f(n) & \text{if } n \geq d \end{cases}$$

The Master Theorem:

1. if $f(n)$ is $O(n^{\log_b a - \varepsilon})$, then $T(n)$ is $\Theta(n^{\log_b a})$
2. if $f(n)$ is $\Theta(n^{\log_b a} \log^k n)$, then $T(n)$ is $\Theta(n^{\log_b a} \log^{k+1} n)$
3. if $f(n)$ is $\Omega(n^{\log_b a + \varepsilon})$, then $T(n)$ is $\Theta(f(n))$,
provided $af(n/b) \leq \delta f(n)$ for some $\delta < 1$.

Example:

$$T(n) = 4T(n/2) + n$$

Solution: $\log_b a = 2$, so case 1 says $T(n)$ is $\Theta(n^2)$.

Master Method: Ex. 2

The form:
$$T(n) = \begin{cases} c & \text{if } n < d \\ aT(n/b) + f(n) & \text{if } n \geq d \end{cases}$$

The Master Theorem:

1. if $f(n)$ is $O(n^{\log_b a - \varepsilon})$, then $T(n)$ is $\Theta(n^{\log_b a})$
2. if $f(n)$ is $\Theta(n^{\log_b a} \log^k n)$, then $T(n)$ is $\Theta(n^{\log_b a} \log^{k+1} n)$
3. if $f(n)$ is $\Omega(n^{\log_b a + \varepsilon})$, then $T(n)$ is $\Theta(f(n))$,
provided $af(n/b) \leq \delta f(n)$ for some $\delta < 1$.

Example:

$$T(n) = 2T(n/2) + n \log n$$

Solution: $\log_b a = 1$, so case 2 says $T(n)$ is $\Theta(n \log^2 n)$.

Master Method: Ex. 3

The form:
$$T(n) = \begin{cases} c & \text{if } n < d \\ aT(n/b) + f(n) & \text{if } n \geq d \end{cases}$$

The Master Theorem:

1. if $f(n)$ is $O(n^{\log_b a - \varepsilon})$, then $T(n)$ is $\Theta(n^{\log_b a})$
2. if $f(n)$ is $\Theta(n^{\log_b a} \log^k n)$, then $T(n)$ is $\Theta(n^{\log_b a} \log^{k+1} n)$
3. if $f(n)$ is $\Omega(n^{\log_b a + \varepsilon})$, then $T(n)$ is $\Theta(f(n))$,
provided $af(n/b) \leq \delta f(n)$ for some $\delta < 1$.

Example:

$$T(n) = T(n/3) + n \log n$$

Solution: $\log_b a = 0$, so case 3 says $T(n)$ is $\Theta(n \log n)$.

Master Method: Ex. 4

The form:
$$T(n) = \begin{cases} c & \text{if } n < d \\ aT(n/b) + f(n) & \text{if } n \geq d \end{cases}$$

The Master Theorem:

1. if $f(n)$ is $O(n^{\log_b a - \varepsilon})$, then $T(n)$ is $\Theta(n^{\log_b a})$
2. if $f(n)$ is $\Theta(n^{\log_b a} \log^k n)$, then $T(n)$ is $\Theta(n^{\log_b a} \log^{k+1} n)$
3. if $f(n)$ is $\Omega(n^{\log_b a + \varepsilon})$, then $T(n)$ is $\Theta(f(n))$,
provided $af(n/b) \leq \delta f(n)$ for some $\delta < 1$.

Example:

$$T(n) = 8T(n/2) + n^2$$

Solution: $\log_b a = 3$, so case 1 says $T(n)$ is $\Theta(n^3)$.

Master Method: Ex. 5

The form:
$$T(n) = \begin{cases} c & \text{if } n < d \\ aT(n/b) + f(n) & \text{if } n \geq d \end{cases}$$

The Master Theorem:

1. if $f(n)$ is $O(n^{\log_b a - \varepsilon})$, then $T(n)$ is $\Theta(n^{\log_b a})$
2. if $f(n)$ is $\Theta(n^{\log_b a} \log^k n)$, then $T(n)$ is $\Theta(n^{\log_b a} \log^{k+1} n)$
3. if $f(n)$ is $\Omega(n^{\log_b a + \varepsilon})$, then $T(n)$ is $\Theta(f(n))$,
provided $af(n/b) \leq \delta f(n)$ for some $\delta < 1$.

Example:

$$T(n) = 9T(n/3) + n^3$$

Solution: $\log_b a = 2$, so case 3 says $T(n)$ is $\Theta(n^3)$.

Master Method: Ex. 6

The form:
$$T(n) = \begin{cases} c & \text{if } n < d \\ aT(n/b) + f(n) & \text{if } n \geq d \end{cases}$$

The Master Theorem:

1. if $f(n)$ is $O(n^{\log_b a - \varepsilon})$, then $T(n)$ is $\Theta(n^{\log_b a})$
2. if $f(n)$ is $\Theta(n^{\log_b a} \log^k n)$, then $T(n)$ is $\Theta(n^{\log_b a} \log^{k+1} n)$
3. if $f(n)$ is $\Omega(n^{\log_b a + \varepsilon})$, then $T(n)$ is $\Theta(f(n))$,
provided $af(n/b) \leq \delta f(n)$ for some $\delta < 1$.

Example:

$$T(n) = T(n/2) + 1 \quad (\text{binary search})$$

Solution: $\log_b a = 0$, so case 2 says $T(n)$ is $\Theta(\log n)$.

Master Method: Ex. 7

The form:
$$T(n) = \begin{cases} c & \text{if } n < d \\ aT(n/b) + f(n) & \text{if } n \geq d \end{cases}$$

The Master Theorem:

1. if $f(n)$ is $O(n^{\log_b a - \varepsilon})$, then $T(n)$ is $\Theta(n^{\log_b a})$
2. if $f(n)$ is $\Theta(n^{\log_b a} \log^k n)$, then $T(n)$ is $\Theta(n^{\log_b a} \log^{k+1} n)$
3. if $f(n)$ is $\Omega(n^{\log_b a + \varepsilon})$, then $T(n)$ is $\Theta(f(n))$,
provided $af(n/b) \leq \delta f(n)$ for some $\delta < 1$.

Example:

$$T(n) = 2T(n/2) + \log n \quad (\text{heap construction})$$

Solution: $\log_b a = 1$, so case 1 says $T(n)$ is $\Theta(n)$.

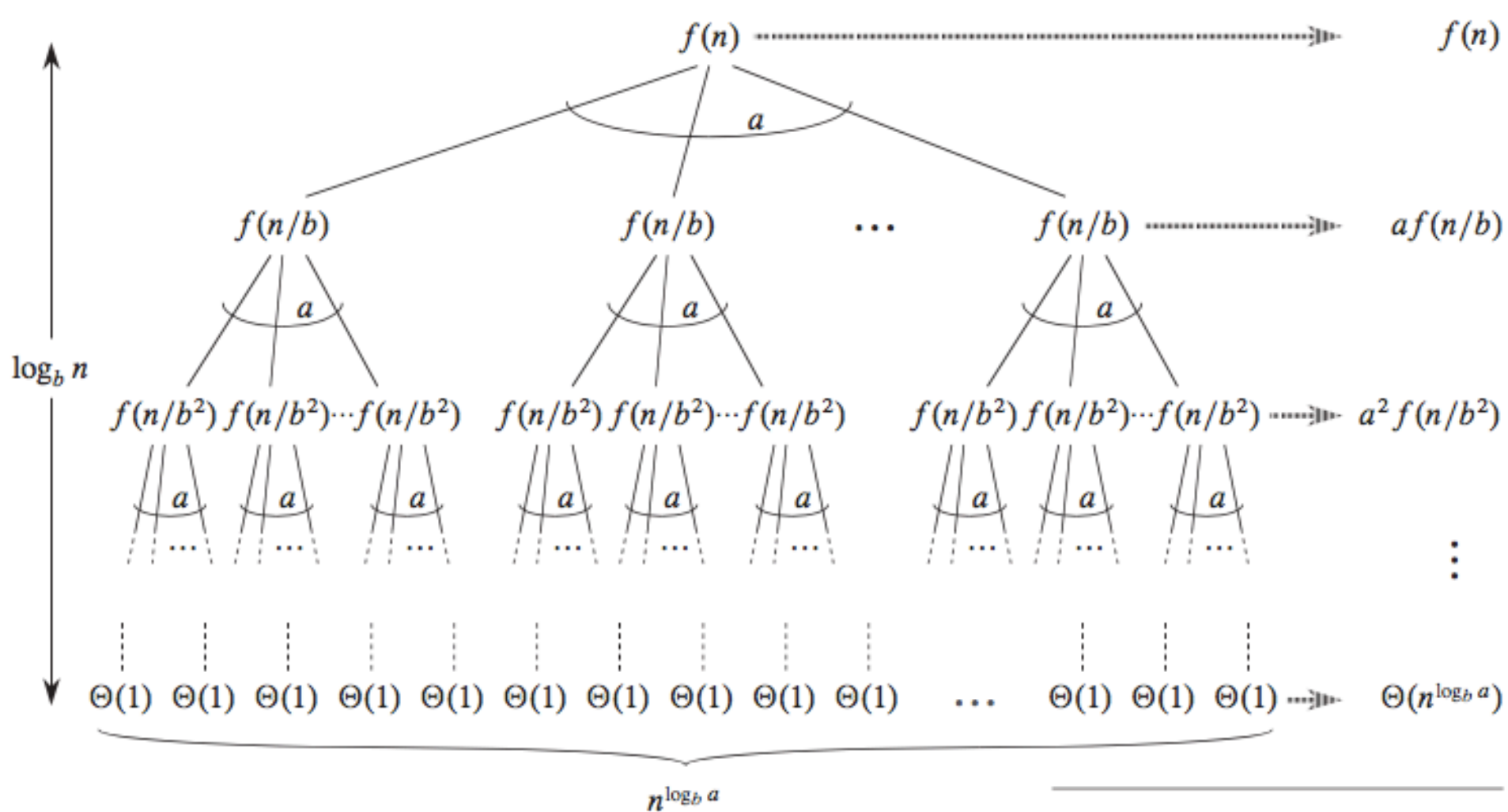
Iterative Justification of the Master Theorem

Use iterative substitution to find a pattern:

$$\begin{aligned}T(n) &= aT(n/b) + f(n) \\&= a(aT(n/b^2)) + f(n/b) + f(n) \\&= a^2T(n/b^2) + af(n/b) + f(n) \\&= a^3T(n/b^3) + a^2f(n/b^2) + af(n/b) + f(n) \\&= \dots \\&= a^{\log_b n}T(1) + \sum_{i=0}^{(\log_b n)-1} a^i f(n/b^i) \\&= n^{\log_b a}T(1) + \sum_{i=0}^{(\log_b n)-1} a^i f(n/b^i)\end{aligned}$$

We then distinguish the three cases as

- Case 1: The first term is dominant
- Case 2: Each part of the summation is equally dominant
- Case 3: The second term is dominant



We then distinguish the three cases as

- Case 1: The first term is dominant
- Case 2: Each part of the summation is equally dominant
- Case 3: The second term is dominant

$$\text{Total: } \Theta(n^{\log_b a}) + \sum_{j=0}^{\log_b n - 1} a^j f(n/b^j)$$

Integer Multiplication

Algorithm: Multiply two n-bit integers I and J.

- Divide step: Split I and J into high-order and low-order bits

$$I = I_h 2^{n/2} + I_l$$

$$J = J_h 2^{n/2} + J_l$$

- We can then define $I * J$ by multiplying the parts and adding:

$$\begin{aligned} I * J &= (I_h 2^{n/2} + I_l) * (J_h 2^{n/2} + J_l) \\ &= I_h J_h 2^n + I_h J_l 2^{n/2} + I_l J_h 2^{n/2} + I_l J_l \end{aligned}$$

- So, $T(n) = 4T(n/2) + n$, which implies $T(n)$ is $\Theta(n^2)$.
- But that is no better than the algorithm we learned in grade school.

Improved Integer Multiplication

Algorithm: Multiply two n-bit integers I and J.

- Divide step: Split I and J into high-order and low-order bits

$$I = I_h 2^{n/2} + I_l$$

$$J = J_h 2^{n/2} + J_l$$

- Observe that there is a different way to multiply parts:

$$\begin{aligned} I * J &= I_h J_h 2^n + [(I_h - I_l)(J_l - J_h) + I_h J_h + I_l J_l] 2^{n/2} + I_l J_l \\ &= I_h J_h 2^n + [(I_h J_l - I_l J_l - I_h J_h + I_l J_h) + I_h J_h + I_l J_l] 2^{n/2} + I_l J_l \\ &= I_h J_h 2^n + (I_h J_l + I_l J_h) 2^{n/2} + I_l J_l \end{aligned}$$

- So, $T(n) = 3T(n/2) + n$, which implies $T(n)$ is $\Theta(n^{\log_2 3})$.
- Thus, $T(n)$ is $O(n^{1.585})$.