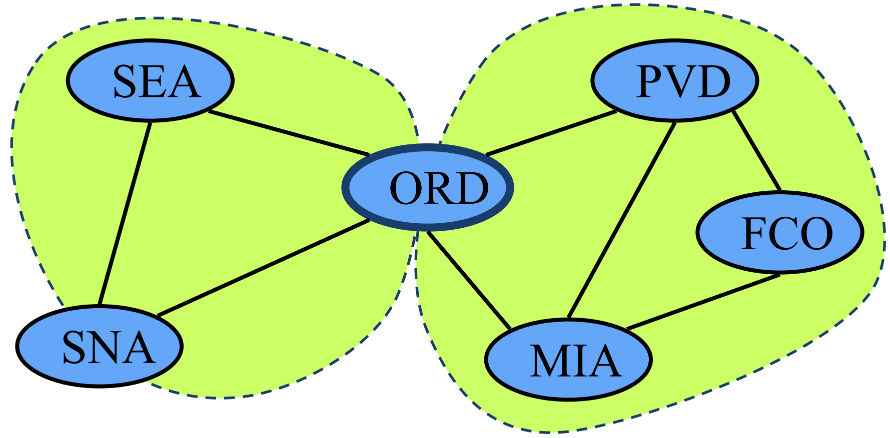
Biconnectivity



Outline and Reading

Definitions (6.3.2)

- Separation vertices and edges
- Biconnected graph
- Biconnected components
- Equivalence classes
- Linked edges and link components

Algorithms (6.3.2)

- Auxiliary graph
- Proxy graph

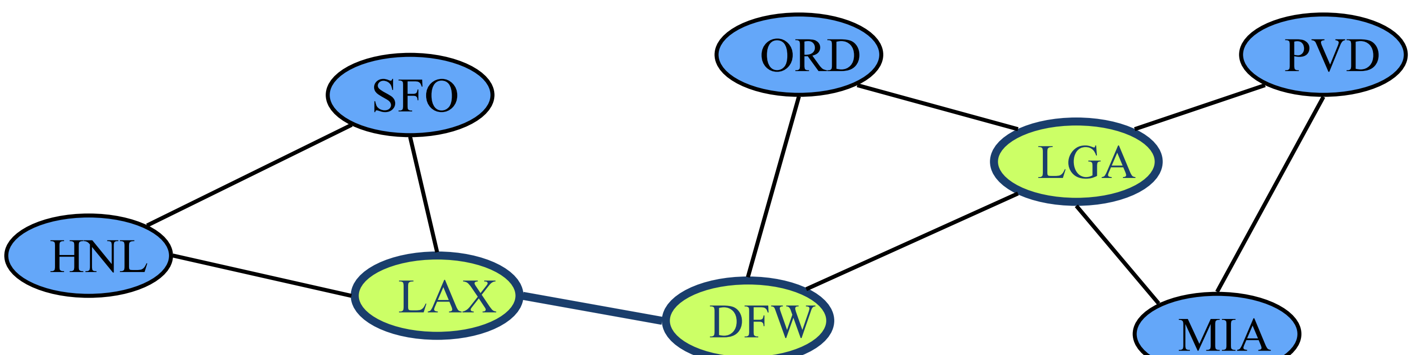
Separation Edges and Vertices

Let ***G*** be a connected graph

- A separation edge of ***G*** is an edge whose removal disconnects ***G.***
  Ex: (DFW,LAX) is a separation edge
- A separation vertex of ***G*** is a vertex whose removal disconnects ***G.***
  Ex: DFW, LGA and LAX are separation vertices

Applications:

- Separation edges and vertices represent single points of failure in a network and are critical to the operation of the network.

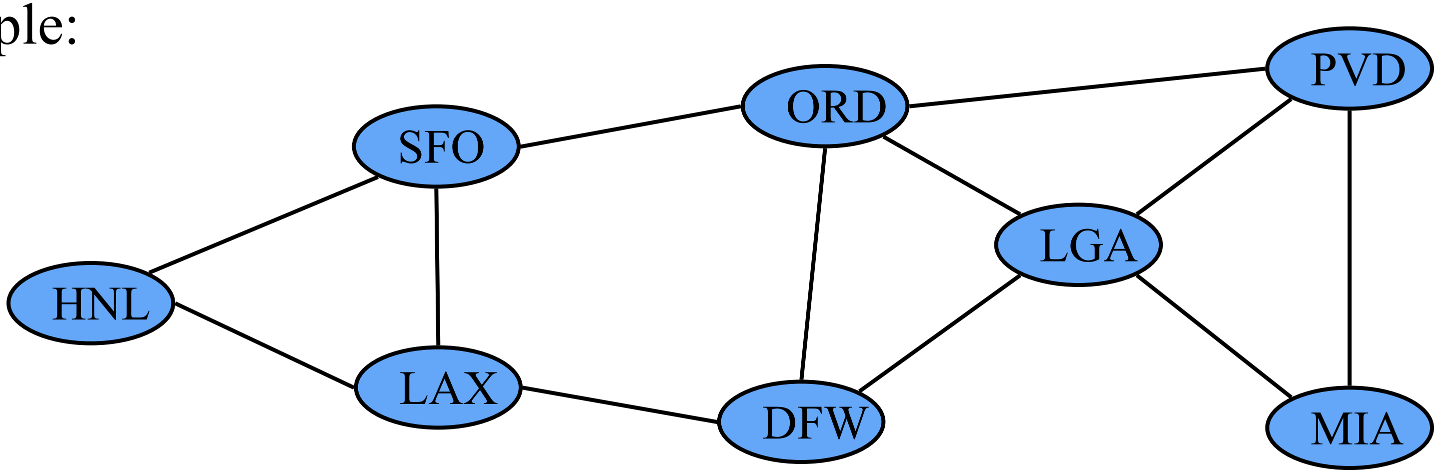


Biconnected Graph

Equivalent definitions of a biconnected graph ***G*:**

- Graph ***G*** has no separation edges and no separation vertices.
- For any two vertices ***u*** and ***v*** of ***G***, there are two disjoint simple paths between ***u*** and ***v*** (i.e., two simple paths between ***u*** and ***v*** that share no other vertices or edges).
- For any two vertices ***u*** and ***v*** of ***G***, there is a simple cycle containing ***u*** and ***v*.**

Example:

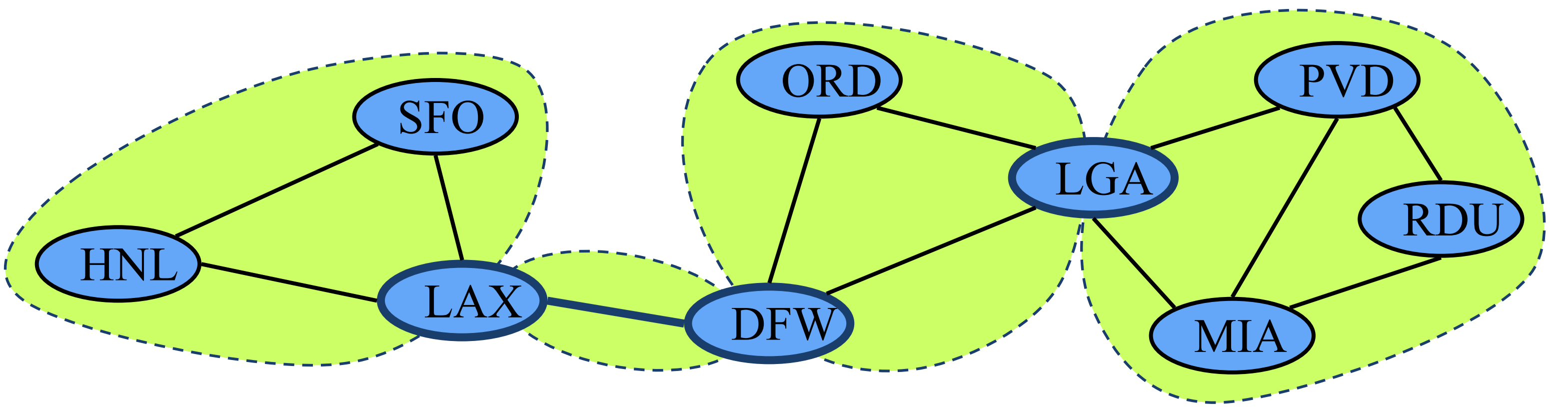


Biconnected Components

- Biconnected component of a graph ***G***
  - A maximal biconnected subgraph of ***G***, or
  - A subgraph consisting of a separation edge of ***G*** and its end vertices

- Interaction of biconnected components
  - An edge belongs to exactly one biconnected component
  - A nonseparation vertex belongs to exactly one biconnected component
  - A separation vertex belongs to two or more biconnected components

- Example of a graph with four biconnected components:



Equivalence Classes

Given a set $\boldsymbol{S}$, a relation $\boldsymbol{R}$ on $\boldsymbol{S}$ is a set of ordered pairs of elements of $\boldsymbol{S}$, i.e., $\boldsymbol{R}$ is a subset of $\boldsymbol{S} \times \boldsymbol{S}$

- An equivalence relation $\boldsymbol{R}$ on $\boldsymbol{S}$ satisfies the following properties
  - Reflexive: $\boldsymbol{R}(\boldsymbol{x},\boldsymbol{x})$ is true for each $\boldsymbol{x}$
  - Symmetric: $\boldsymbol{R}(\boldsymbol{x},\boldsymbol{y}) = \boldsymbol{R}(\boldsymbol{y},\boldsymbol{x})$ for each $\boldsymbol{x},\boldsymbol{y}$
  - Transitive: $\boldsymbol{R}(\boldsymbol{x},\boldsymbol{y}) \wedge \boldsymbol{R}(\boldsymbol{y},\boldsymbol{z}) \longrightarrow \boldsymbol{R}(\boldsymbol{x},\boldsymbol{z})$ for each $\boldsymbol{x},\boldsymbol{y},\boldsymbol{z}$
- An equivalence relation $\boldsymbol{R}$ on $\boldsymbol{S}$ induces a partition of the elements of $\boldsymbol{S}$ into equivalence classes

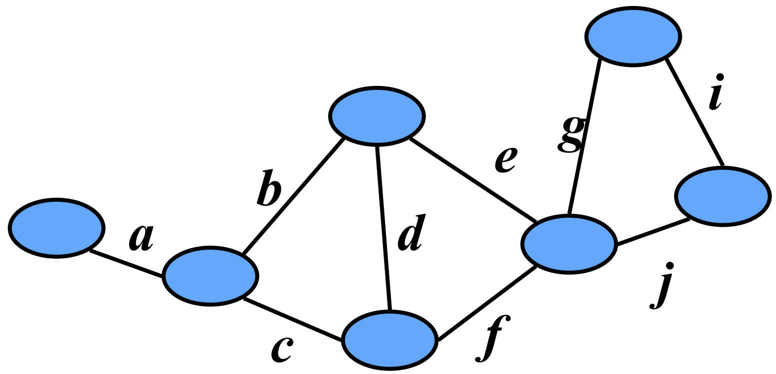
Example (connectivity relation among the vertices of a graph):

- Let $\boldsymbol{V}$ be the set of vertices of a graph $\boldsymbol{G}$
- Define the relation
  $\boldsymbol{C} = \{(\boldsymbol{v},\boldsymbol{w}) \in \boldsymbol{V} \times \boldsymbol{V}$ such that $\boldsymbol{G}$ has a path from $\boldsymbol{v}$ to $\boldsymbol{w}\}$
- Relation $\boldsymbol{C}$ is an equivalence relation
- The equivalence classes of relation $\boldsymbol{C}$ are the vertices in each connected component of graph $\boldsymbol{G}$

Link Relation

Edges ***e*** and ***f*** of connected graph ***G*** are **linked** if

- ***e*** = ***f***, or
- ***G*** has a simple cycle containing ***e*** and ***f***

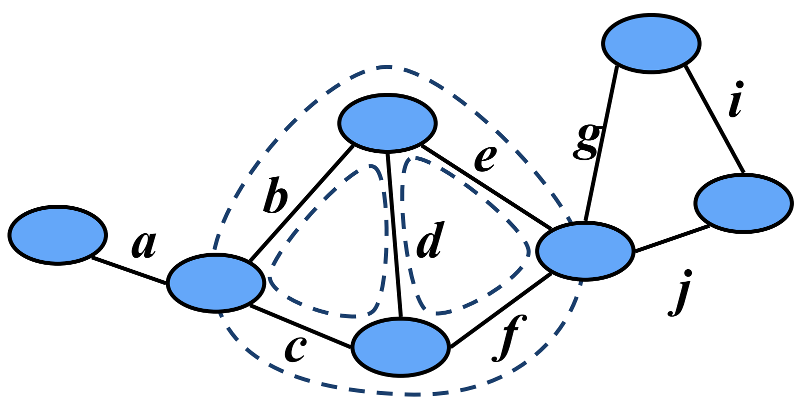


Equivalence classes of linked edges:
{***a***} {***b***, ***c***, ***d***, ***e***, ***f***} {***g***, ***i***, ***j***}

Theorem: The link relation on the edges of a graph is an equivalence relation.

Proof Sketch:

- The reflexive and symmetric properties follow from the definition
- For the transitive property, consider two simple cycles sharing an edge

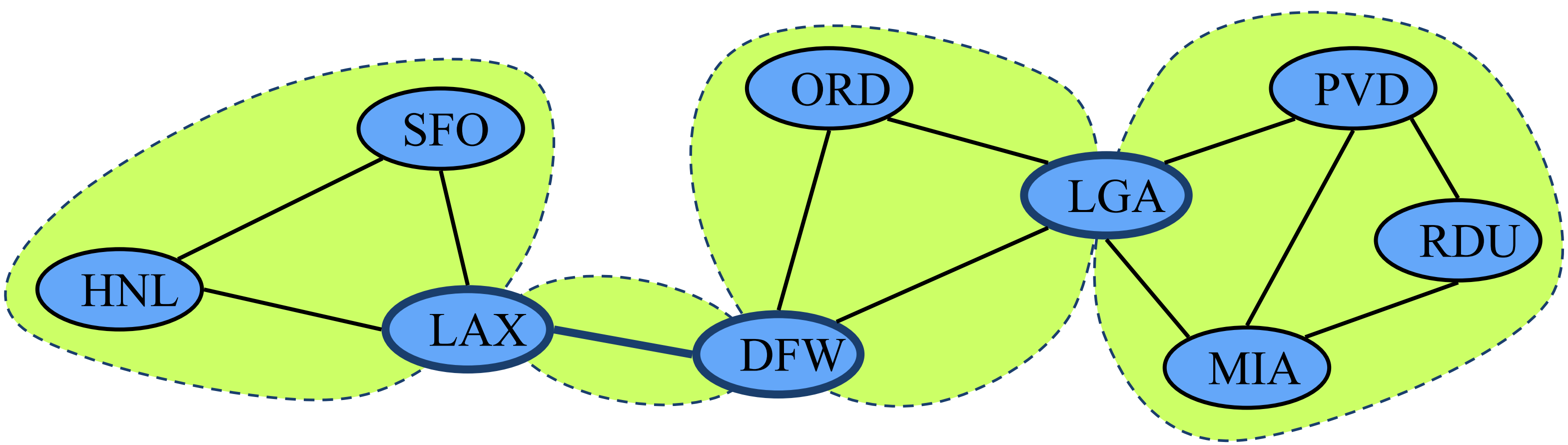


Link Components

The link components of a connected graph G are the equivalence classes of edges with respect to the link relation

A biconnected component of G is the subgraph of G induced by an equivalence class of linked edges

- A separation edge is a single-element equivalence class of linked edges
- A separation vertex has incident edges in at least two distinct equivalence classes of linked edge

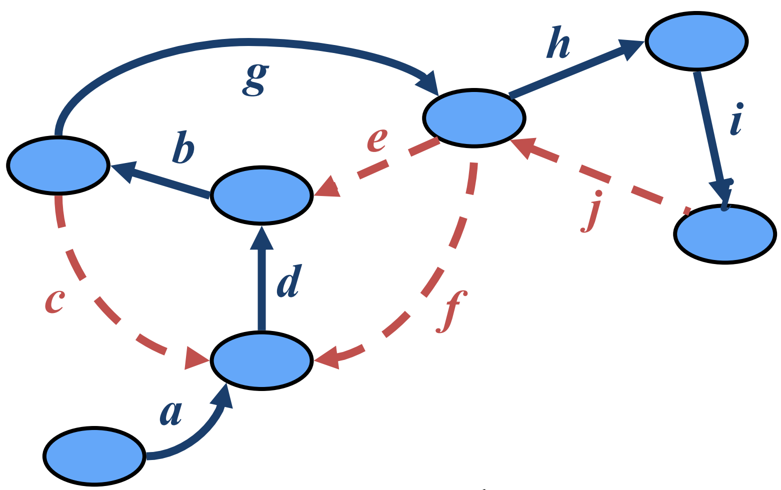


Auxiliary Graph

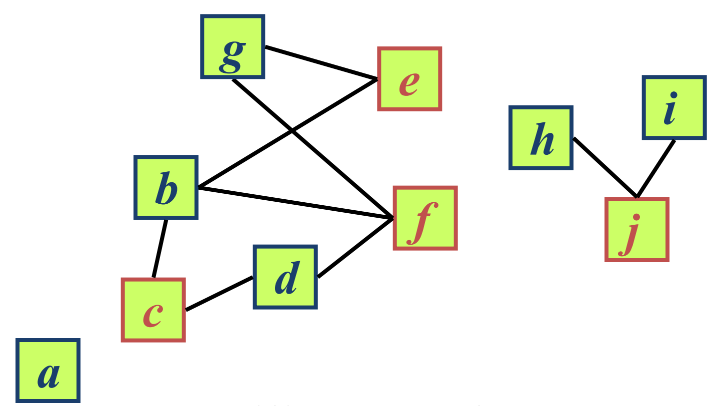
Auxiliary graph ***B*** for a connected graph ***G***

- Associated with a DFS traversal of ***G***
- The vertices of ***B*** are the edges of ***G***
- For each back edge ***e*** of ***G***, ***B*** has edges (e,f_1), (e,f_2) , …, (e,f_k), where $f_1, f_2, \ldots, f_k$ are the discovery edges of ***G*** that form a simple cycle with ***e***

The connected components of B correspond to the link components of ***G***



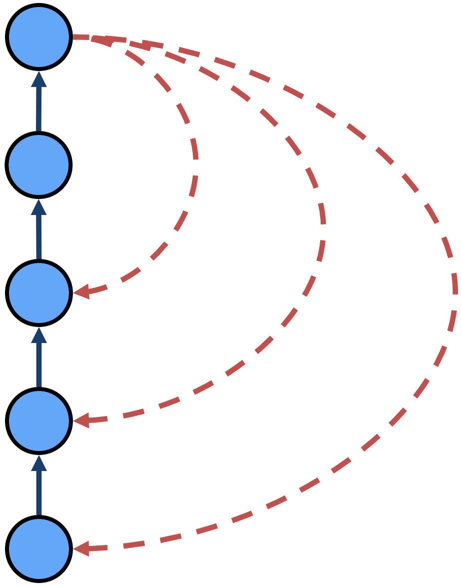
DFS on graph ***G***



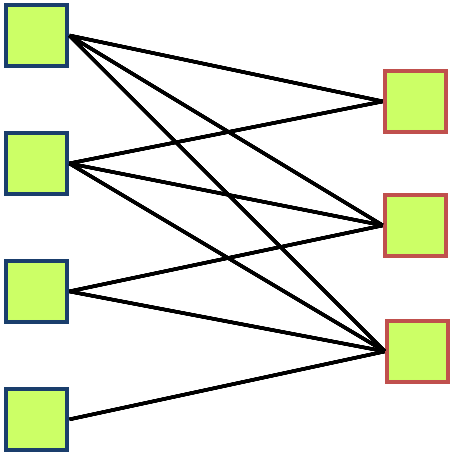
Auxiliary graph ***B***

Auxiliary Graph (cont.)

In the worst case, the number of edges of the auxiliary graph is proportional to ***nm*****.**



DFS on graph ***G***



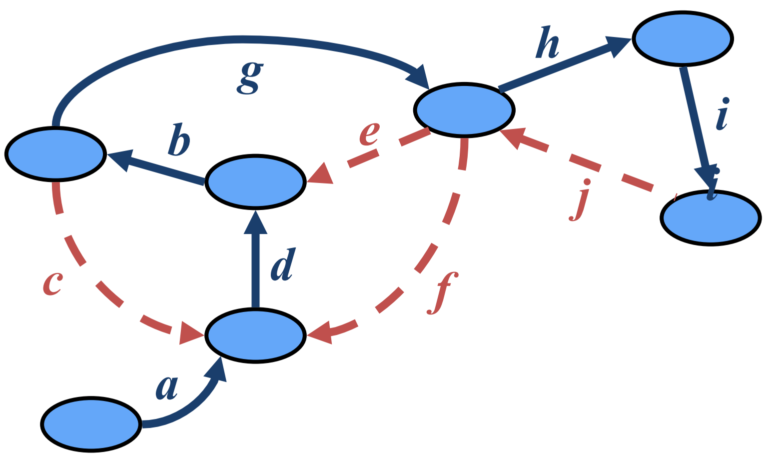
Auxiliary graph ***B***

An Algorithm to Compute Biconnected Components

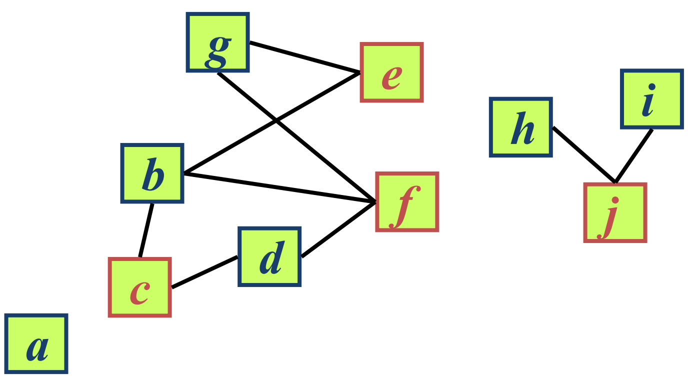
1. Perform DFS traversal on G
2. Compute auxiliary graph B
3. Compute connected components of B
4. For each connected component of B, output vertices of B (edges of G) as a link component of G

Running time is $O(nm)$. Why?

Can we do better?



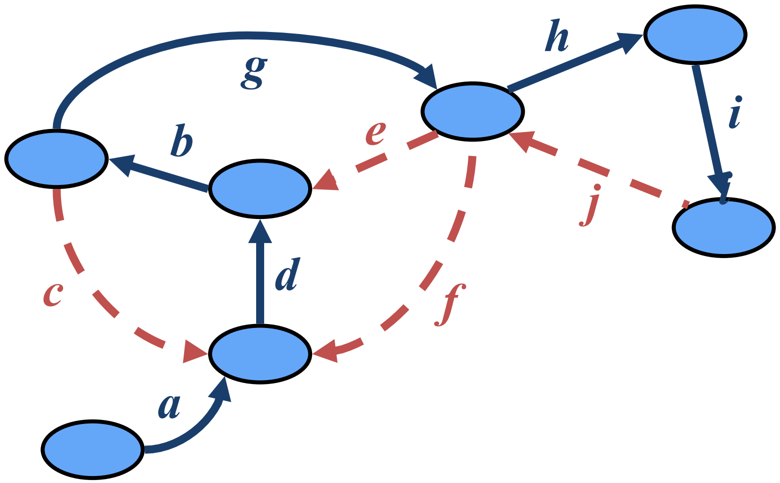
DFS on graph ***G***



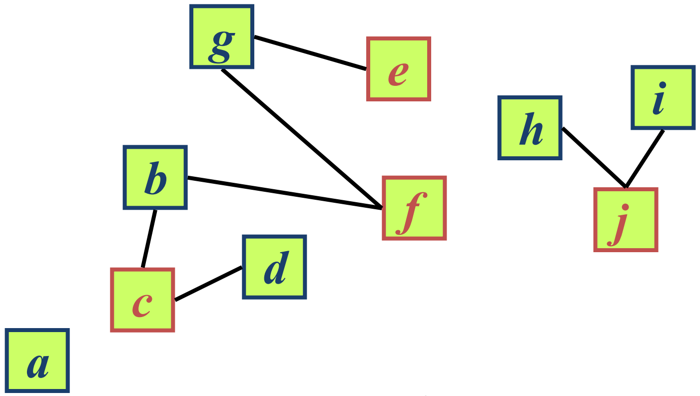
Auxiliary graph ***B***

Proxy Graph

```
Algorithm proxyGraph(G)
  Input connected graph G
  Output proxy graph F for G
  F ← empty graph
  DFS(G, s) { s is any vertex of G}
  for all discovery edges e of G
    F.insertVertex(e)
    setLabel(e, UNLINKED)
  for all vertices v of G in DFS visit order
    for all back edges e = (u,v)
      F.insertVertex(e)
      repeat  {add edges to F only as necessary}
        f ← discovery edge with dest. u
        F.insertEdge(e,f,∅)
        if getLabel(f) = UNLINKED
          setLabel(f, LINKED)
          u ← origin of edge f
        else
          u ← v { ends the loop }
      until u = v
  return F
```



DFS on graph G



Proxy graph F

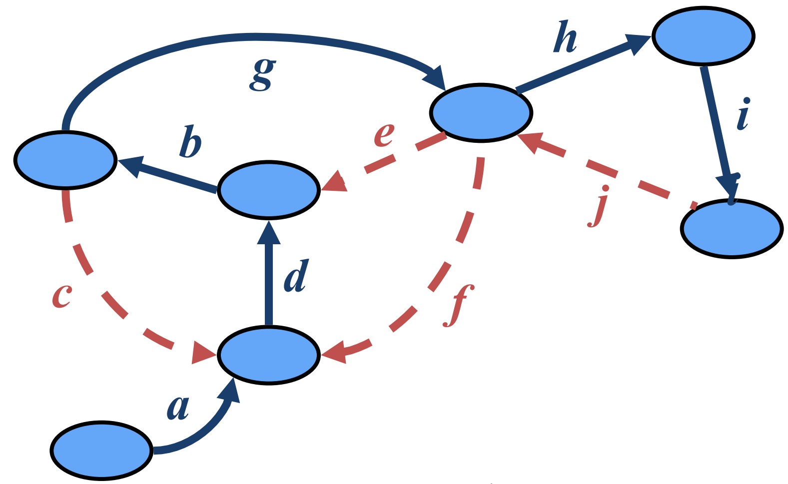
Proxy Graph (cont.)

Proxy graph ***F*** for a connected graph ***G***

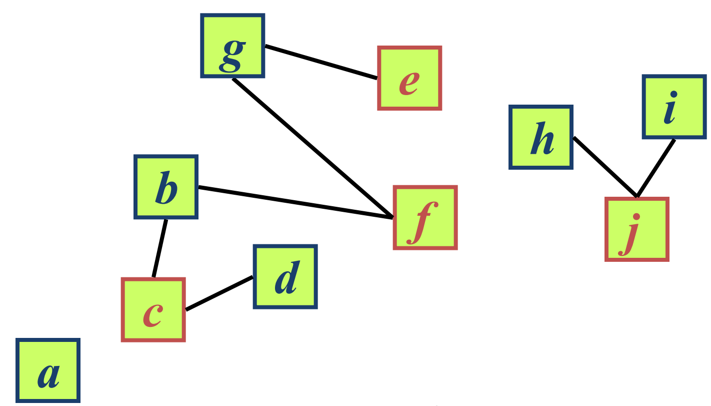
- Spanning forest of the auxiliary graph ***B***
- Has ***m*** vertices and ***O***(***m***) edges
- Can be constructed in ***O***(***n*** + ***m***) time
- Its connected components (trees) correspond to the link components of ***G***

Given a graph ***G*** with ***n*** vertices and ***m*** edges, we can compute the following in ***O***(***n***+***m***) time

- The biconnected components of ***G***
- The separation vertices of ***G***
- The separation edges of ***G***



DFS on graph ***G***



Proxy graph ***F***