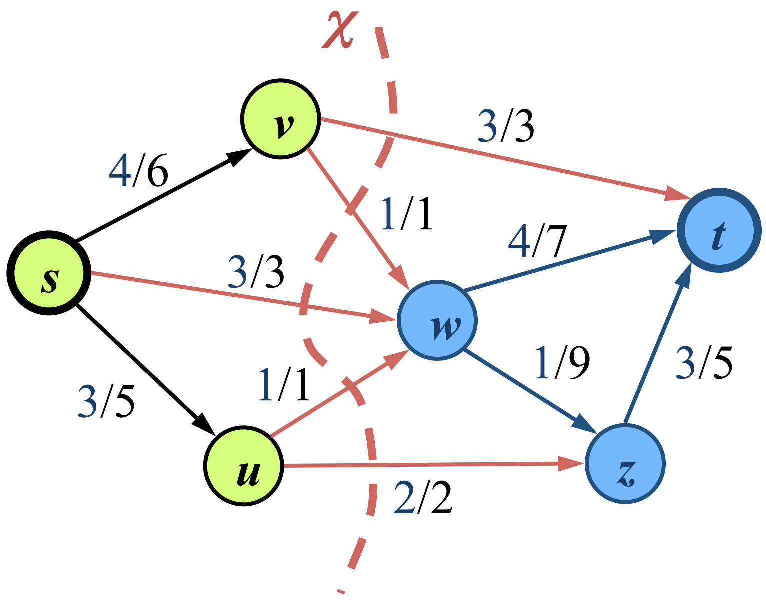
Maximum Flow



Outline and Reading

Flow networks

- Flow (8.1.1)
- Cut (8.1.2)

Maximum flow

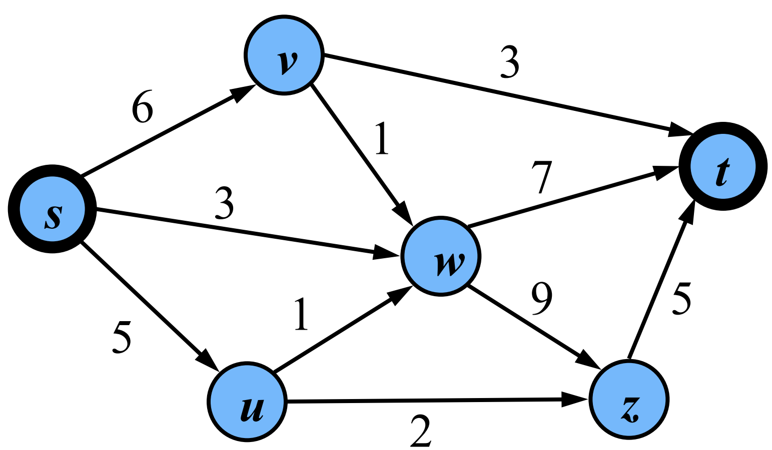
- Augmenting path (8.2.1)
- Maximum flow and minimum cut (8.2.1)
- Ford-Fulkerson’s algorithm (8.2.2-8.2.3)
- Edmond Karp’s algorithm (8.2.4)

Flow Network

A flow network (or just network) N consists of

- A weighted digraph G with nonnegative integer edge weights, where the weight of an edge e is called the capacity $c(e)$ of e
- Two distinguished vertices, s and t of G, called the source and sink, respectively, such that s has no incoming edges and t has no outgoing edges.

Example:



Flow

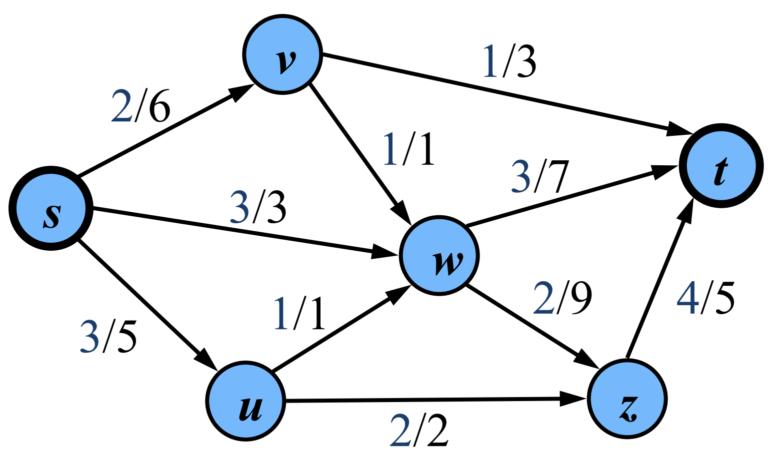
A flow $\boldsymbol{f}$ for a network $\boldsymbol{N}$ is is an assignment of an integer values $\boldsymbol{f(e)}$ to each edge $\boldsymbol{e}$ that satisfies the following properties:

- Capacity rule: for each edge $\boldsymbol{e}$, $0 \le \boldsymbol{f}(\boldsymbol{e}) \le \boldsymbol{c}(\boldsymbol{e})$
- Conservation rule: for each vertex $\boldsymbol{v} \neq \boldsymbol{s}, \boldsymbol{t}$ $$\sum_{e \in E^-(v)} f(e) = \sum_{e \in E^+(v)} f(e)$$

  where $\boldsymbol{E}^-(\boldsymbol{v})$ and $\boldsymbol{E}^+(\boldsymbol{v})$ are the incoming and outgoing edges of $\boldsymbol{v}$, resp.

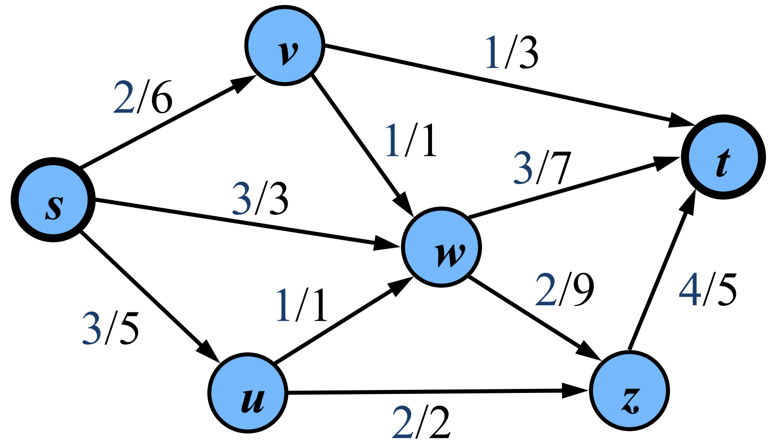
- The value of a flow $\boldsymbol{f}$, denoted $|\boldsymbol{f}|$, is the total flow from the source, which is the same as the total flow into the sink

Example:

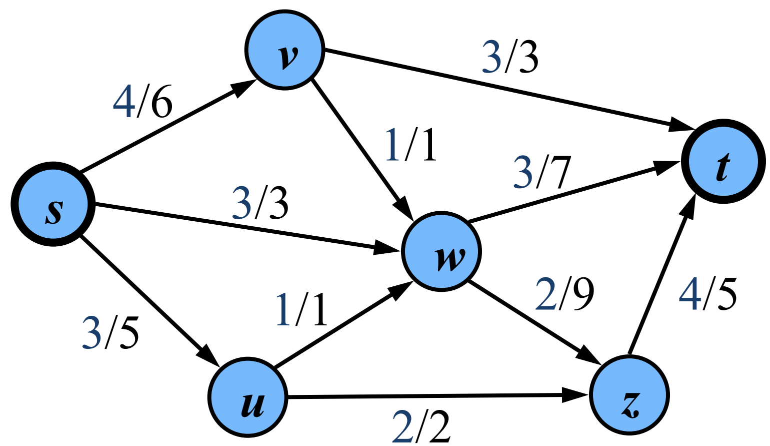


Maximum Flow

- A flow for a network N is said to be maximum if its value is the largest of all flows for N
- The maximum flow problem consists of finding a maximum flow for a given network N
- Applications
  - Hydraulic systems
  - Electrical circuits
  - Traffic movements
  - Freight transportation



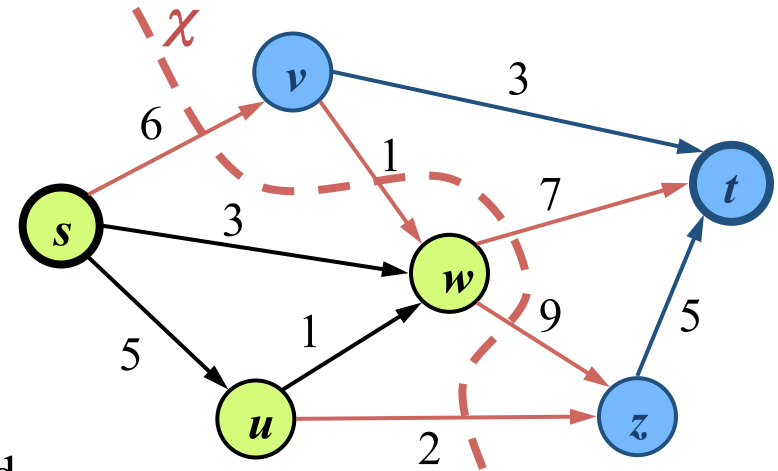
Flow of value 8 = 2 + 3 + 3 = 1 + 3 + 4



Maximum flow of value 10 = 4 + 3 + 3 = 3 + 3 + 4

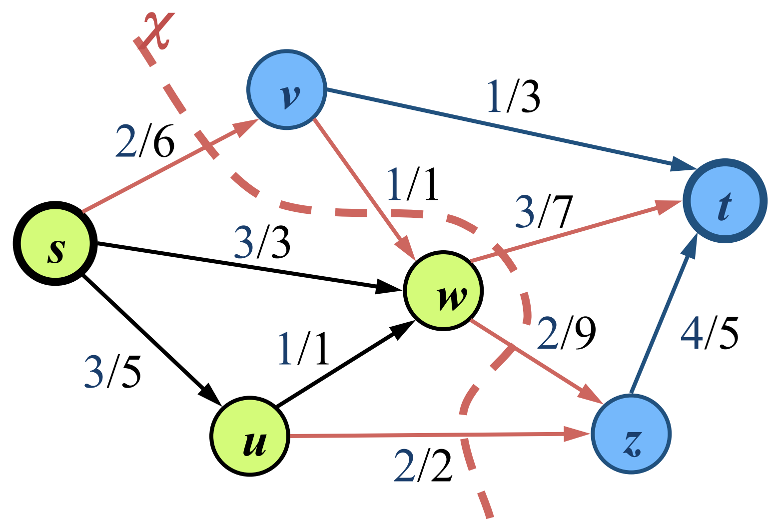
Cut

- A cut of a network N with source s and sink t is a partition $\chi = (V_s, V_t)$ of the vertices of N such that $s \in V_s$ and $t \in V_t$
  - Forward edge of cut χ: origin in V_s and destination in V_t
  - Backward edge of cut χ: origin in V_t and destination in V_s



- Flow $f(\chi)$ across a cut χ: total flow of forward edges minus total flow of backward edges

- Capacity $c(\chi)$ of a cut χ: total capacity of forward edges



- Example:
  - $c(\chi) = 24$
  - $f(\chi) = 8$

Flow and Cut

Lemma:

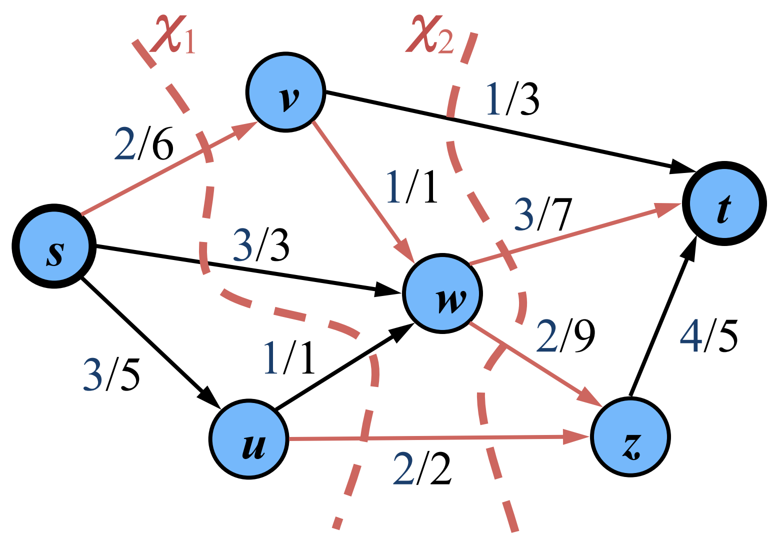
The flow $f(\chi)$ across any cut χ is equal to the flow value $|f|$

Lemma:

The flow $f(\chi)$ across a cut χ is less than or equal to the capacity $c(\chi)$ of the cut

Theorem:

The value of any flow is less than or equal to the capacity of any cut, i.e., for any flow f and any cut χ, we have $|f| \le c(\chi)$

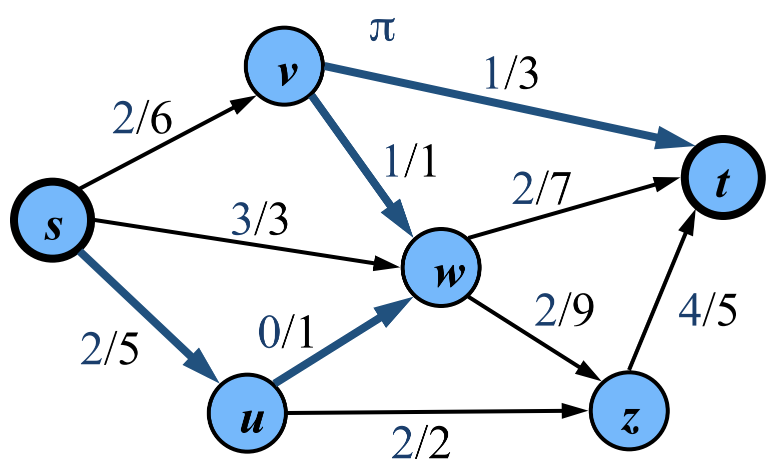


$c(\chi_1) = 12 = 6 + 3 + 1 + 2$
$c(\chi_2) = 21 = 3 + 7 + 9 + 2$
$|f| = 8$

Augmenting Path

Consider a flow f for a network N

- Let e be an edge from u to v:
  - Residual capacity of e from u to v: $\Delta_f(u, v) = c(e) - f(e)$
  - Residual capacity of e from v to u: $\Delta_f(v, u) = f(e)$



$\Delta_f(s,u) = 3$
$\Delta_f(u,w) = 1$
$\Delta_f(w,v) = 1$
$\Delta_f(v,t) = 2$
$\Delta_f(\pi) = 1$
$|f| = 7$

- Let π be a path from s to t
  - The residual capacity $\Delta_f(\pi)$ of π is the smallest of the residual capacities of the edges of π in the direction from s to t

A path π from s to t is an augmenting path if $\Delta_f(\pi) > 0$

Flow Augmentation

Lemma:

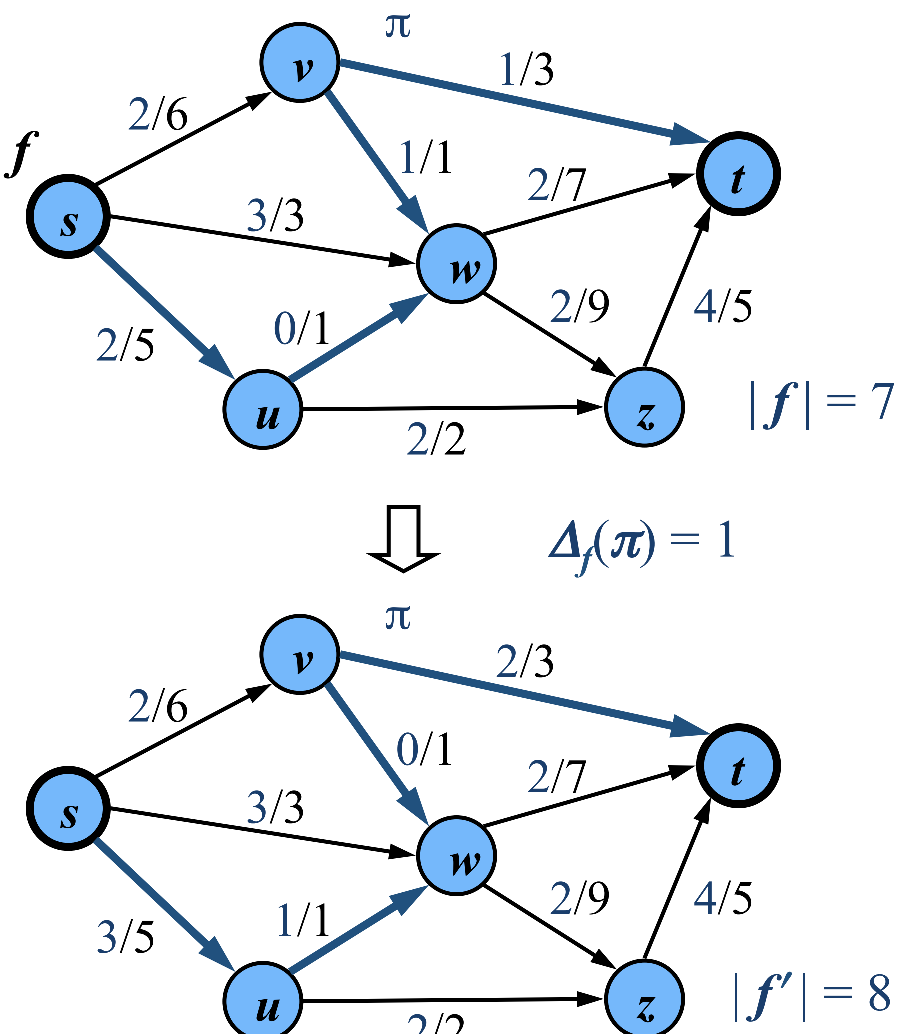
Let $\boldsymbol{\pi}$ be an augmenting path for flow $\boldsymbol{f}$ in network $\boldsymbol{N}$. There exists a flow $\boldsymbol{f'}$ for $\boldsymbol{N}$ of value

$$|\boldsymbol{f'}| = |\boldsymbol{f}| + \Delta_f(\boldsymbol{\pi})$$

Proof:

We compute flow $\boldsymbol{f'}$ by modifying the flow on the edges of $\boldsymbol{\pi}$

- Forward edge:
$$\boldsymbol{f'}(\boldsymbol{e}) = \boldsymbol{f}(\boldsymbol{e}) + \Delta_f(\boldsymbol{\pi})$$
- Backward edge:
$$\boldsymbol{f'}(\boldsymbol{e}) = \boldsymbol{f}(\boldsymbol{e}) - \Delta_f(\boldsymbol{\pi})$$



Ford-Fulkerson's Algorithm

- Initially, $f(e) = 0$ for each edge e
- Repeatedly
  - Search for an augmenting path π
  - Augment by $\Delta_f(\pi)$ the flow along the edges of π
- A specialization of DFS (or BFS) searches for an augmenting path
  - An edge e is traversed from u to v provided $\Delta_f(u, v) > 0$

```
Algorithm FordFulkersonMaxFlow(N)
  for all e ∈ G.edges()
    setFlow(e, 0)
  while G has an augmenting path π
    { compute residual capacity Δ of π }
    Δ ← ∞
    for all edges e ∈ π
      { compute residual capacity δ of e }
      if e is a forward edge of π
        δ ← getCapacity(e) − getFlow(e)
      else { e is a backward edge }
        δ ← getFlow(e)
      if δ < Δ
        Δ ← δ
    { augment flow along π }
    for all edges e ∈ π
      if e is a forward edge of π
        setFlow(e, getFlow(e) + Δ)
      else { e is a backward edge }
        setFlow(e, getFlow(e) − Δ)
```

Max-Flow and Min-Cut

- Termination of Ford-Fulkerson's algorithm
  - There is no augmenting path from $\boldsymbol{s}$ to $\boldsymbol{t}$ with respect to the current flow $\boldsymbol{f}$
- Define

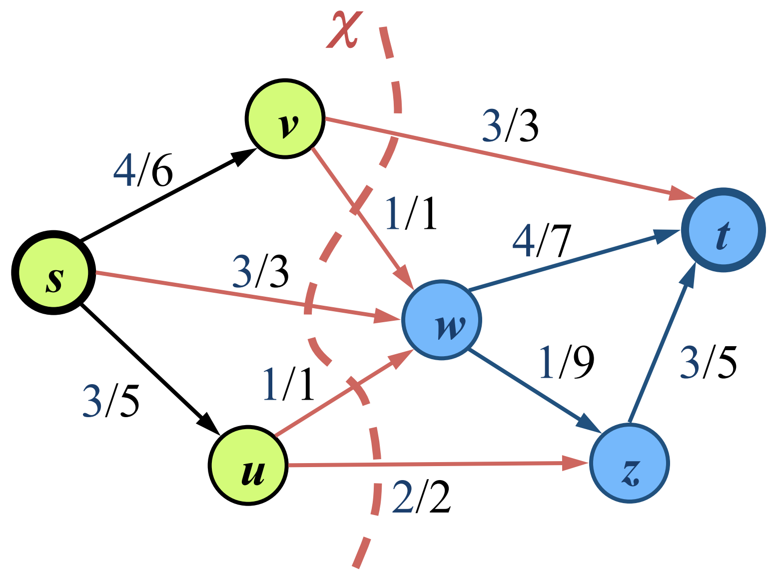
  V_s set of vertices reachable from $\boldsymbol{s}$ by augmenting paths

  V_t set of remaining vertices
- Cut $\chi = (V_s, V_t)$ has capacity

  $c(\chi) = |f|$
  - Forward edge: $f(e) = c(e)$
  - Backward edge: $f(e) = 0$
- Thus, flow f has maximum value and cut χ has minimum capacity

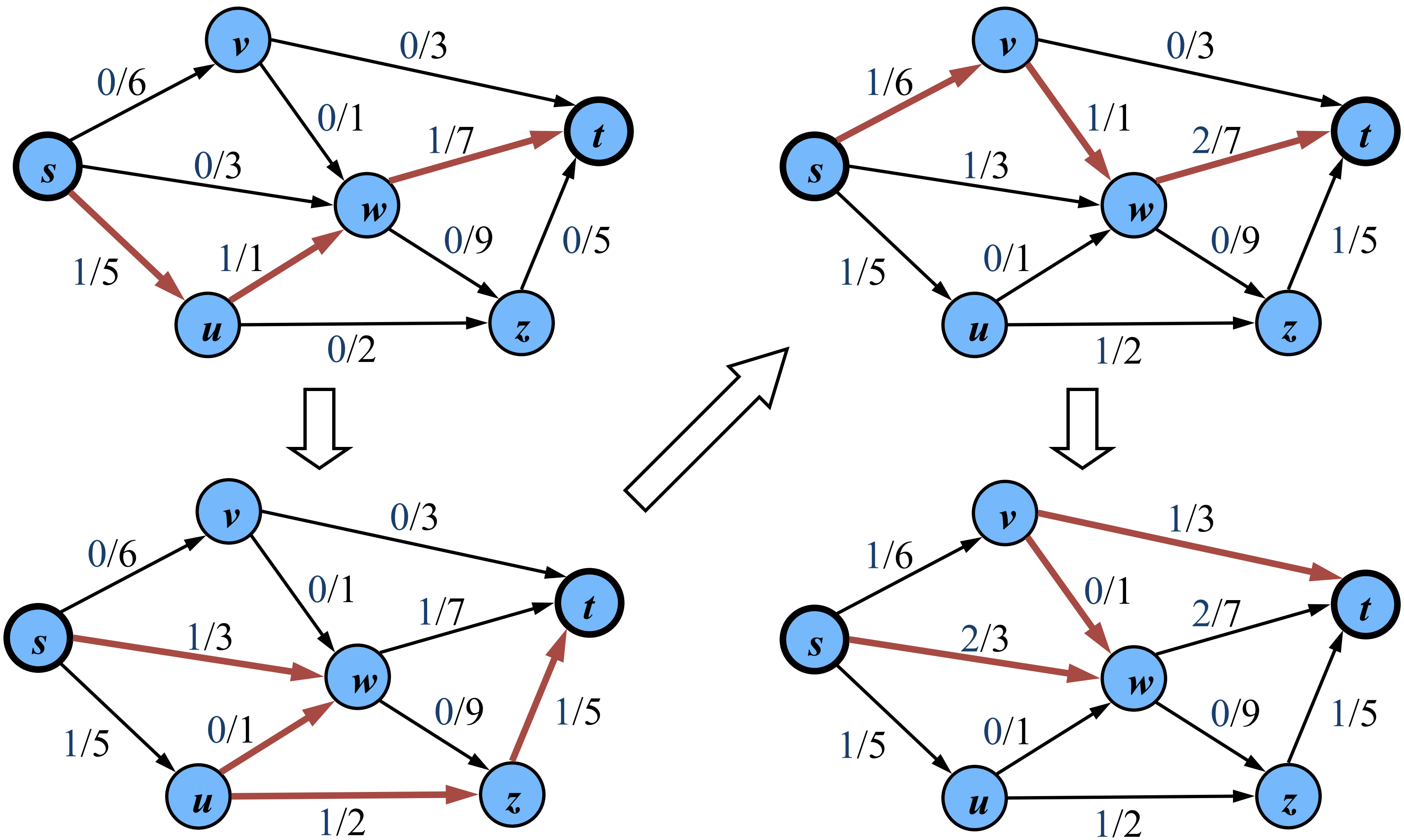
Theorem:

The value of a maximum flow is equal to the capacity of a minimum cut



$c(\chi) = |f| = 10$

Example (1)



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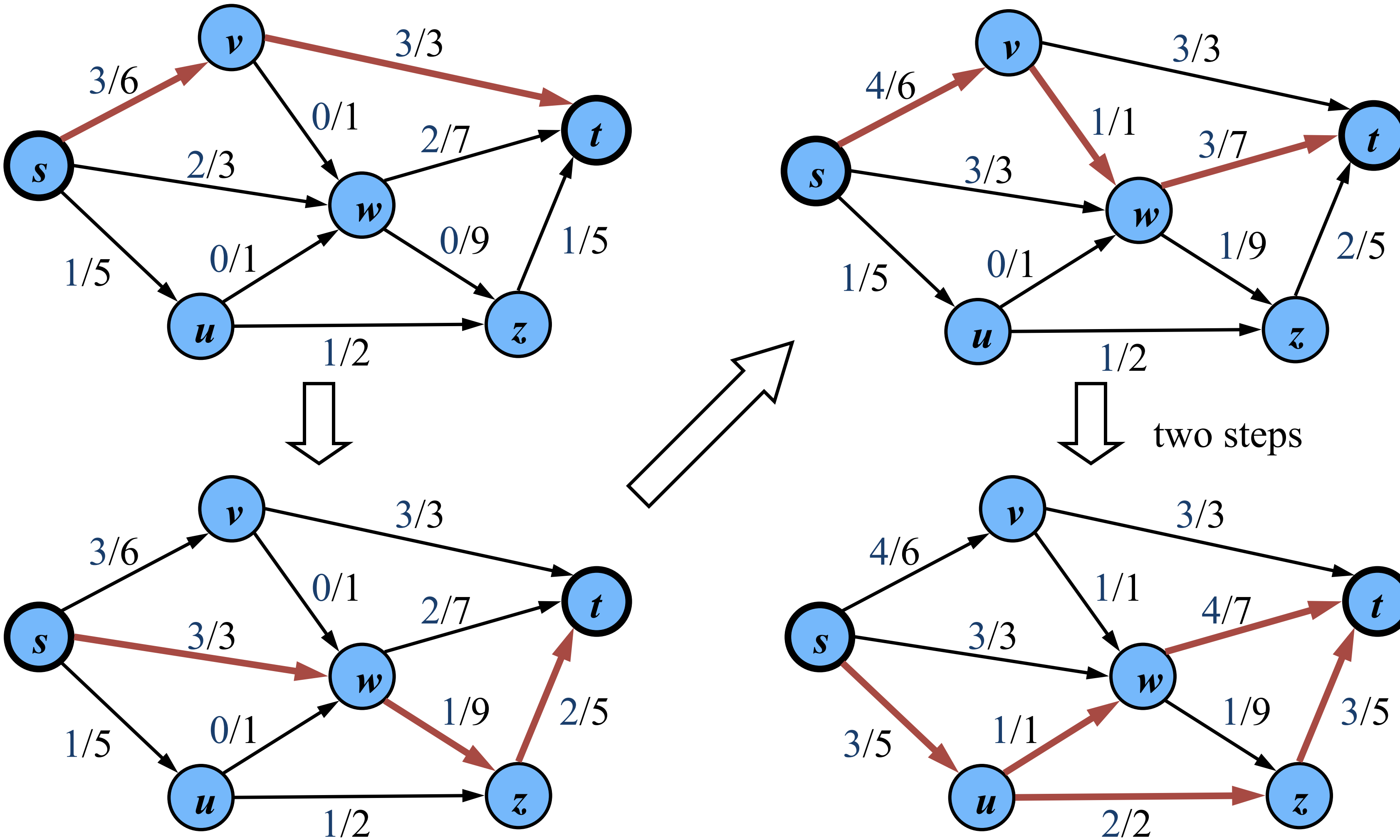
Example (2)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two steps
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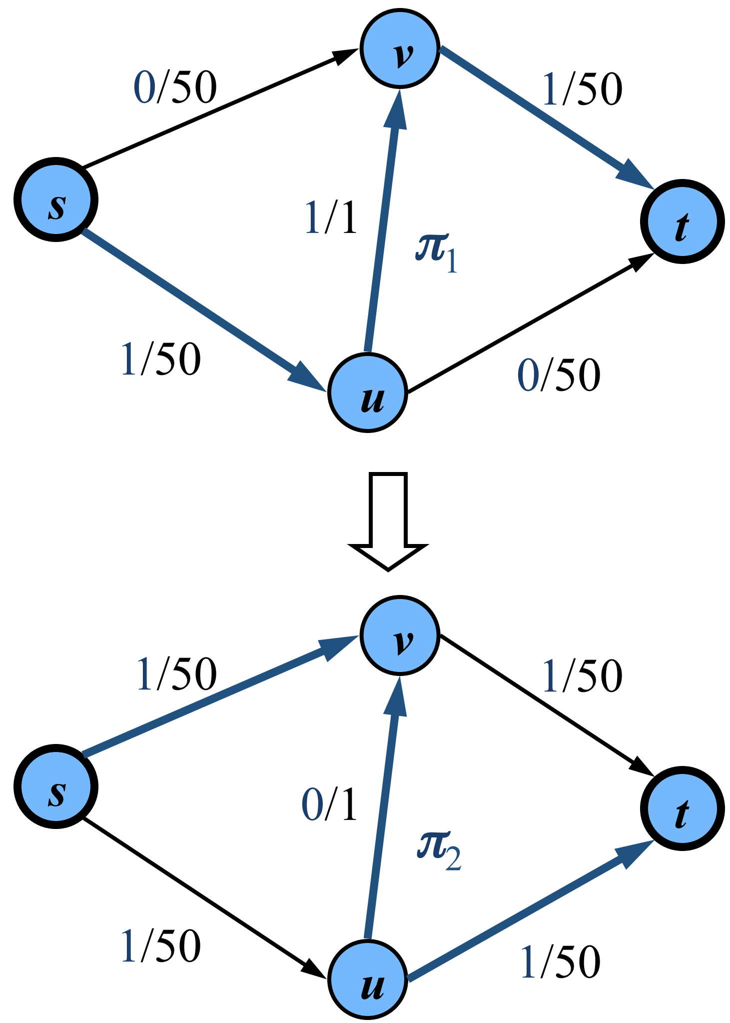
Analysis

- In the worst case, Ford-Fulkerson's algorithm performs $|f^*|$ flow augmentations, where f^* is a maximum flow
- Example
  - The augmenting paths found alternate between π_1 and π_2
  - The algorithm performs 100 augmentations
- Finding an augmenting path and augmenting the flow takes $O(n + m)$ time
- The running time of Ford-Fulkerson's algorithm is $O(|f^*|(n + m))$



Edmonds-Karp Algorithm

- A variation of the Ford Fulkerson algorithm that uses BFS to find augmenting paths
- Use a 'more' greedy choice to find good augmenting paths
  - choose an augmenting path with the smallest number of edges
- Running time is $O(nm^2)$ (proof in book)