Dynamic Programming

Outline and Reading

- Matrix Chain-Product (5.3.1)
- The General Technique (5.3.2)
- 0-1 Knapsack Problem (5.3.3)

Matrix Chain Product

Dynamic Programming is a general algorithm design paradigm.

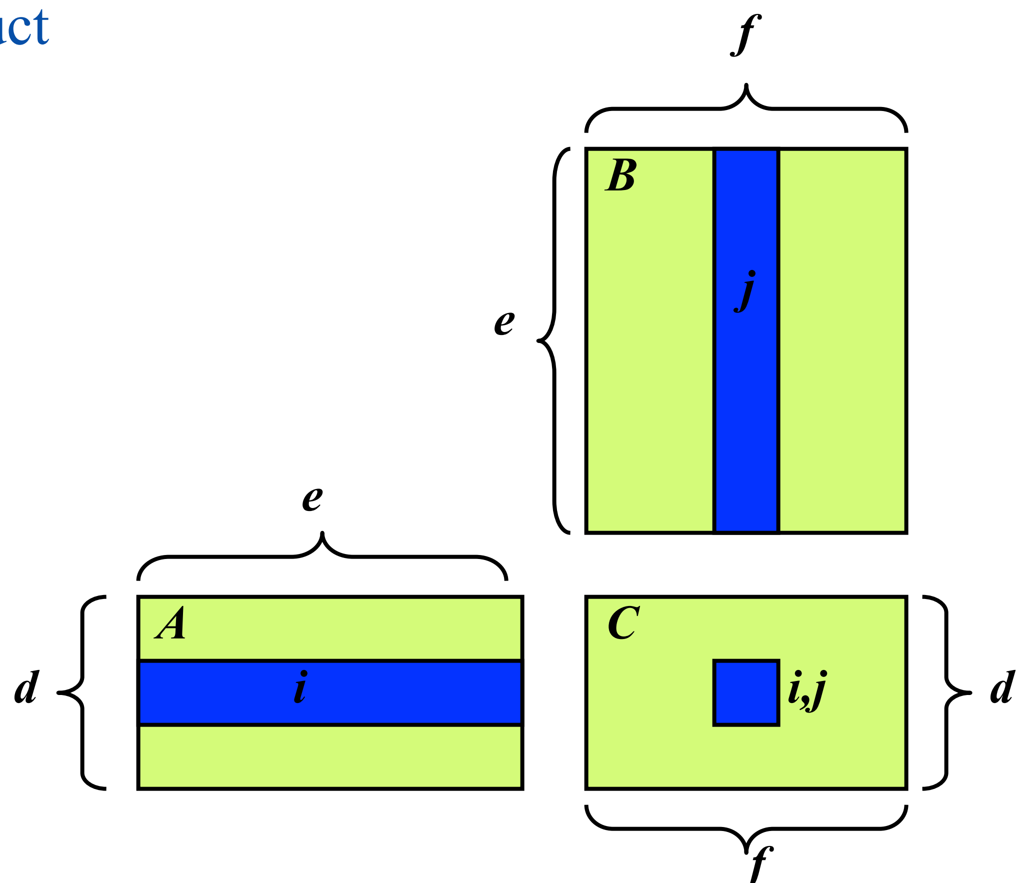
- Rather than give the general structure, we first give a motivating example: Matrix Chain-Product

Review: Matrix Multiplication

- $C = A*B$
- A is $d \times e$ and B is $e \times f$

$$C[i,j] = \sum_{k=0}^{e-1} A[i,k] * B[k,j]$$

- $O(d \cdot e \cdot f)$ time



Matrix Chain Product

Matrix Chain-Product**:**

- Compute $A=A_0*A_1*\ldots*A_{n-1}$
- A_i is $d_i \times d_{i+1}$
- **Problem**: How to parenthesize in such a way that minimizes the total number of scalar multiplications?

Example:

- B is 3×100
- C is 100×5
- D is 5×5
- (B*C)*D takes 1500 + 75 = 1575 ops
- B*(C*D) takes 1500 + 2500 = 4000 ops

One Approach: Brute Force

- Try all possible ways to parenthesize $A=A_0*A_1*\ldots*A_{n-1}$
- Calculate number of operations for each one
- Pick the one that is best

Running time:

- The number of parenthesizations is equal to the number of binary trees with n nodes
  - This is exponential!
  - It is called the Catalan number, and it is almost 4^n.
- This is a terrible algorithm.

Another Approach: Greedy (v1)

Idea: Repeatedly select the product that uses (up) the most operations.

Counter-example:

- A is 10×5
- B is 5×10
- C is 10×5
- D is 5×10

This greedy approach gives (A*B)*(C*D)

- takes 500+1000+500 = 2000 ops

A better solution: A*((B*C)*D)

- takes 500+250+250 = 1000 ops

Another Approach: Greedy (v2)

Idea: Repeatedly select the product that uses the fewest operations.

Counter-example:

- A is 101×11
- B is 11×9
- C is 9×100
- D is 100×99

This greedy approach gives A*((B*C)*D))

- takes 109989+9900+108900=228789 ops

A better solution is (A*B)*(C*D)

- takes 9999+89991+89100=189090 ops

The greedy approach is not giving us the optimal value.

"Recursive" Approach

Define subproblems:

- Find the best parenthesization of $A_i * A_{i+1} * \ldots * A_j$.
- Let $N_{i,j}$ denote the number of operations done by this subproblem.
- The optimal solution for the whole problem is $N_{0,n-1}$.

Subproblem optimality: The optimal solution can be defined in terms of optimal subproblems

- There has to be a final multiplication (root of the expression tree) for the optimal solution.
- Say, the final multiply is at index i: $(A_0 * \ldots * A_i) * (A_{i+1} * \ldots * A_{n-1})$.
- Then the optimal solution $N_{0,n-1}$ is the sum of two optimal subproblems, $N_{0,i}$ and $N_{i+1,n-1}$ plus the time for the last multiply.
- If the global optimum did not have these optimal subproblems, we could define an even better "optimal" solution.

Characterizing Equation

- The global optimal has to be defined in terms of optimal subproblems, depending on where the final multiply is at.

- Consider all possible places for that final multiply:
  - Recall that A_i is a $d_i \times d_{i+1}$ dimensional matrix.
  - So, a characterizing equation for $N_{i,j}$ is the following:

$$N_{i,j} = \min_{i \le k < j} \{ N_{i,k} + N_{k+1,j} + d_i d_{k+1} d_{j+1} \}$$

- Note that subproblems are not independent – meaning subproblems overlap.

Dynamic Programming Algorithm Visualization

$$N_{i,j} = \min_{i \le k < j} \{ N_{i,k} + N_{k+1,j} + d_i d_{k+1} d_{j+1} \}$$

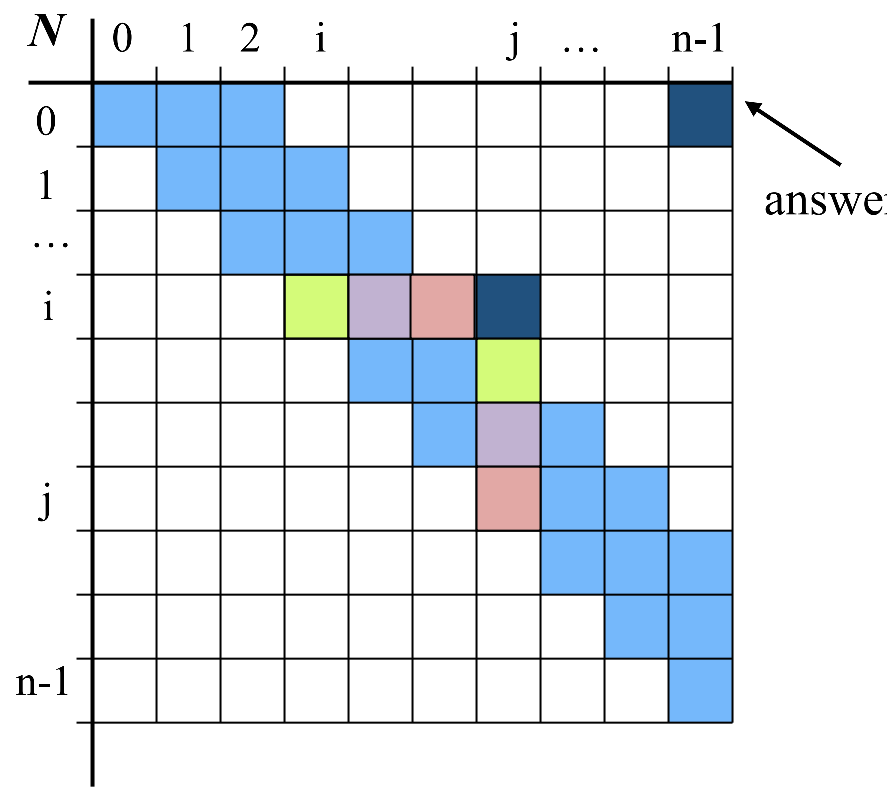
The bottom-up construction fills in the N array by diagonals

$N_{i,j}$ gets values from previous entries in i-th row and j-th column

Filling in each entry in the N table takes O(n) time.

- Total run time: $O(n^3)$

Getting actual parenthesization can be done by remembering "k" for each N entry



Dynamic Programming Algorithm

Since subproblems overlap, we don't use recursion.

Instead, we construct optimal subproblems "bottom-up."

$N_{i,i}$'s are easy, so start with them

Then do problems of "length" 2,3,… subproblems, and so on.

Running time: $O(n^3)$

Algorithm ***matrixChain***(***S***):

Input: sequence ***S*** of ***n*** matrices to be multiplied

Output: number of operations in an optimal parenthesization of ***S***

for $i \leftarrow 0$ **to** $n - 1$ **do**

  $N_{i,i} \leftarrow 0$

for $length \leftarrow 1$ **to** $n - 1$ **do**

  $\{ length = j - i$ is the length of the chain$\}$

  for $i \leftarrow 0$ **to** $n - 1 - length$ **do**

    $j \leftarrow i + length$

    $N_{i,j} \leftarrow +\infty$

    for $k \leftarrow i$ **to** $j - 1$ **do**

      $N_{i,j} \leftarrow \min\{N_{i,j}, N_{i,k} + N_{k+1,j} + d_i d_{k+1} d_{j+1}\}$

      record k that produces minimum $N_{i,j}$

return $N_{0,n-1}$

General
Dynamic Programming Technique

Applies to an optimization problem that at first seems to require a lot of time (possibly exponential), provided we have:

- Simple subproblems**:** the subproblems can be defined in terms of a few variables, such as j, k, l, m, and so on.
- Subproblem overlap**:** the subproblems are not independent, but instead they overlap (hence, should be constructed bottom-up).
- Subproblem optimality**:** the global optimum value can be defined in terms of optimal subproblems

0/1 Knapsack Problem

Given: A set S of n items, with each item i having
- w_i - a positive weight
- b_i - a positive benefit

Goal: Choose items with maximum total benefit but with weight at most W.

If we are not allowed to take fractional amounts, then this is the 0/1 knapsack problem.
- In this case, we let *T* denote the set of items we take
- Objective: maximize

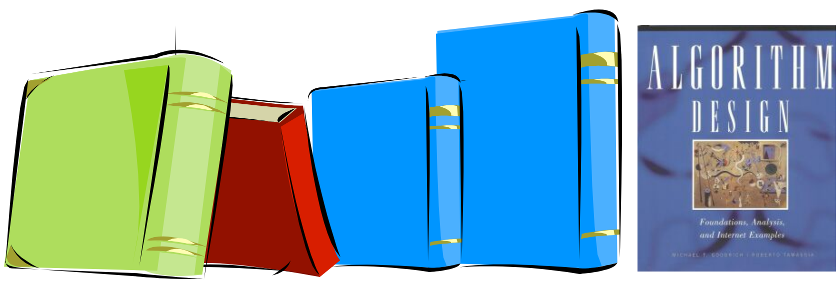
$$\sum_{i \in T} b_i$$

- Constraint:

$$\sum_{i \in T} w_i \leq W$$

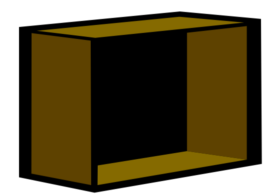
Example

- Given: A set S of n items, with each item i having
  - b_i - a positive "benefit"
  - w_i - a positive "weight"
- Goal: Choose items with maximum total benefit but with weight at most W.



Items:	1	2	3	4	5
Weight:	4 in	2 in	2 in	6 in	2 in
Benefit:	$20	$3	$6	$25	$80

"knapsack"



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Solution:
- item 5 ($80, 2 in)
- item 3 ($6, 2in)
- item 1 ($20, 4in)

A 0/1 Knapsack Algorithm: First Attempt

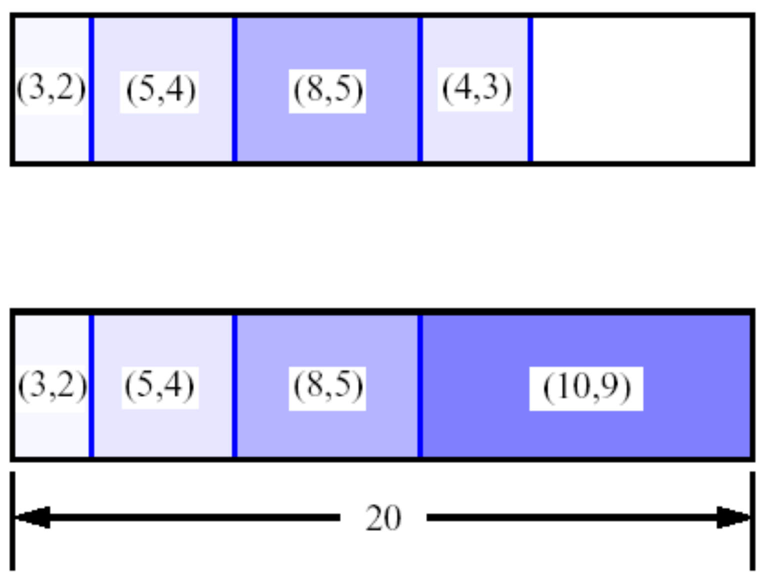
S_k: Set of items numbered 1 to k.

- Idea: Define B[k] = best selection from S_k.

- Problem: does not have subproblem optimality.
  - Consider set S={(3,2),(5,4),(8,5),(4,3),(10,9)} of (benefit, weight) pairs and total weight W = 20

Best for S_4:

Best for S_5:



A 0/1 Knapsack Algorithm: Second Attempt

S_k: Set of items numbered 1 to k.

- Idea: Define B[k,w] to be the best selection from S_k with weight at most w
- Good news: this does have subproblem optimality.

$$B[k,w] = \begin{cases} B[k-1,w] & \text{if } w_k > w \\ \max\{B[k-1,w], B[k-1,w-w_k]+b_k\} & \text{else} \end{cases}$$

That is, the best subset of S_k with weight at most w is either

- the best subset of S_{k-1} with weight at most w or
- the best subset of S_{k-1} with weight at most $w-w_k$ plus item k

0/1 Knapsack Algorithm

$$B[k,w] = \begin{cases} B[k-1,w] & \text{if } w_k > w \\ \max\{B[k-1,w],\ B[k-1,w-w_k]+b_k\} & \text{else} \end{cases}$$

- Recall the definition of B[k,w]
- Since B[k,w] is defined in terms of B[k−1,*], we can use two arrays of instead of a matrix
- Running time: O(nW).
- Not a polynomial-time algorithm since W may be large
- This is a pseudo-polynomial time algorithm

Algorithm ***01Knapsack***(***S, W***):
    Input: set S of n items with benefit b_i and weight w_i; maximum weight W
    Output: benefit of best subset of S with weight at most W
    let A and B be arrays of length $W+1$
    for $w \leftarrow 0$ **to** W **do**
        $B[w] \leftarrow 0$
    for $k \leftarrow 1$ **to** n **do**
        copy array B into array A
        for $w \leftarrow w_k$ **to** W **do**
            if $A[w-w_k] + b_k > A[w]$ **then**
                $B[w] \leftarrow A[w-w_k] + b_k$
    return $B[W]$