Merge Sort

Divide-and-Conquer

Divide-and-conquer is a general algorithm design paradigm:

- Divide: divide the input data S in two disjoint subsets S_1 and S_2
- Recur: solve the subproblems associated with S_1 and S_2
  - the base case for the recursion are subproblems of size 0 or 1
- Conquer: combine the solutions for S_1 and S_2 into a solution for S

Merge-sort is a sorting algorithm based on the divide-and-conquer paradigm

- Like heap-sort
  - Uses a comparator
  - Has $O(n \log n)$ running time
- Unlike heap-sort
  - Does not use an auxiliary priority queue
  - Accesses data in a sequential manner (suitable to sort data on a disk)

Merge Sort

Merge-sort on an input sequence $\boldsymbol{S}$ with $\boldsymbol{n}$ elements consists of three steps:

- Divide: partition $\boldsymbol{S}$ into two sequences $\boldsymbol{S}_1$ and $\boldsymbol{S}_2$ of about $\boldsymbol{n}/2$ elements each
- Recur: recursively sort $\boldsymbol{S}_1$ and $\boldsymbol{S}_2$
- Conquer: merge $\boldsymbol{S}_1$ and $\boldsymbol{S}_2$ into a unique sorted sequence

```
Algorithm mergeSort(S, C)
    Input sequence S with n elements, comparator C
    Output sequence S sorted according to C
    if S.size() > 1
        (S1, S2) ← partition(S, n/2)
        mergeSort(S1, C)
        mergeSort(S2, C)
        S ← merge(S1, S2)
```

Merging two sorted sequences

- The conquer step of merge-sort consists of merging two sorted sequences ***A*** and ***B*** into a sorted sequence ***S*** containing the union of the elements of ***A*** and ***B***

- Merging two sorted sequences, each with $n/2$, takes $O(n)$ time

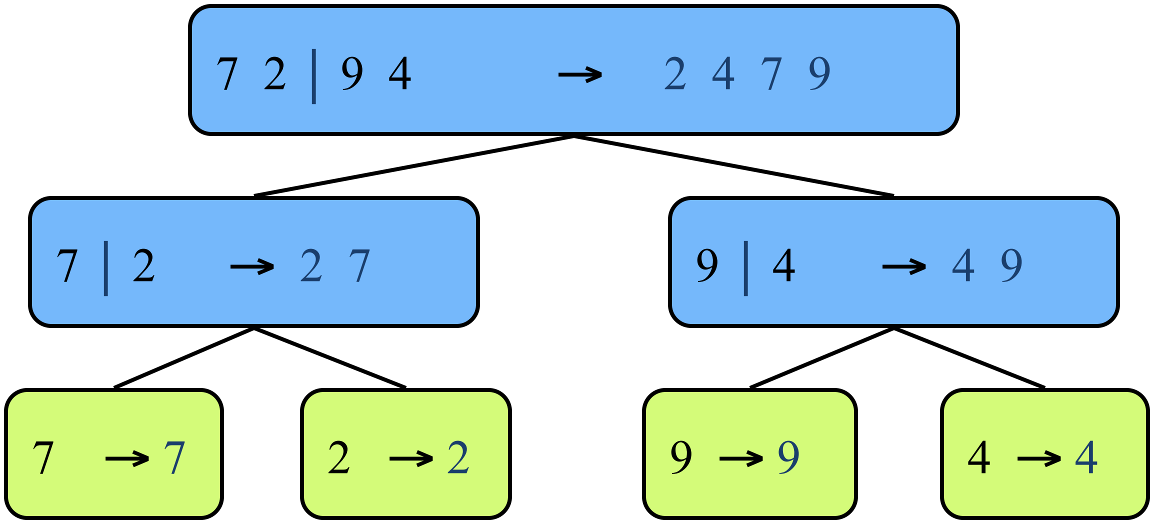
```
Algorithm merge(A, B)
    Input sequences A and B with n/2 elements each
    Output sorted sequence of A ∪ B

    S ← empty sequence
    while ¬A.isEmpty() ∧ ¬B.isEmpty()
        if A.first().element() < B.first().element()
            S.insertLast(A.remove(A.first()))
        else
            S.insertLast(B.remove(B.first()))
    while ¬A.isEmpty()
        S.insertLast(A.remove(A.first()))
    while ¬B.isEmpty()
        S.insertLast(B.remove(B.first()))
    return S
```

Merge-Sort Tree

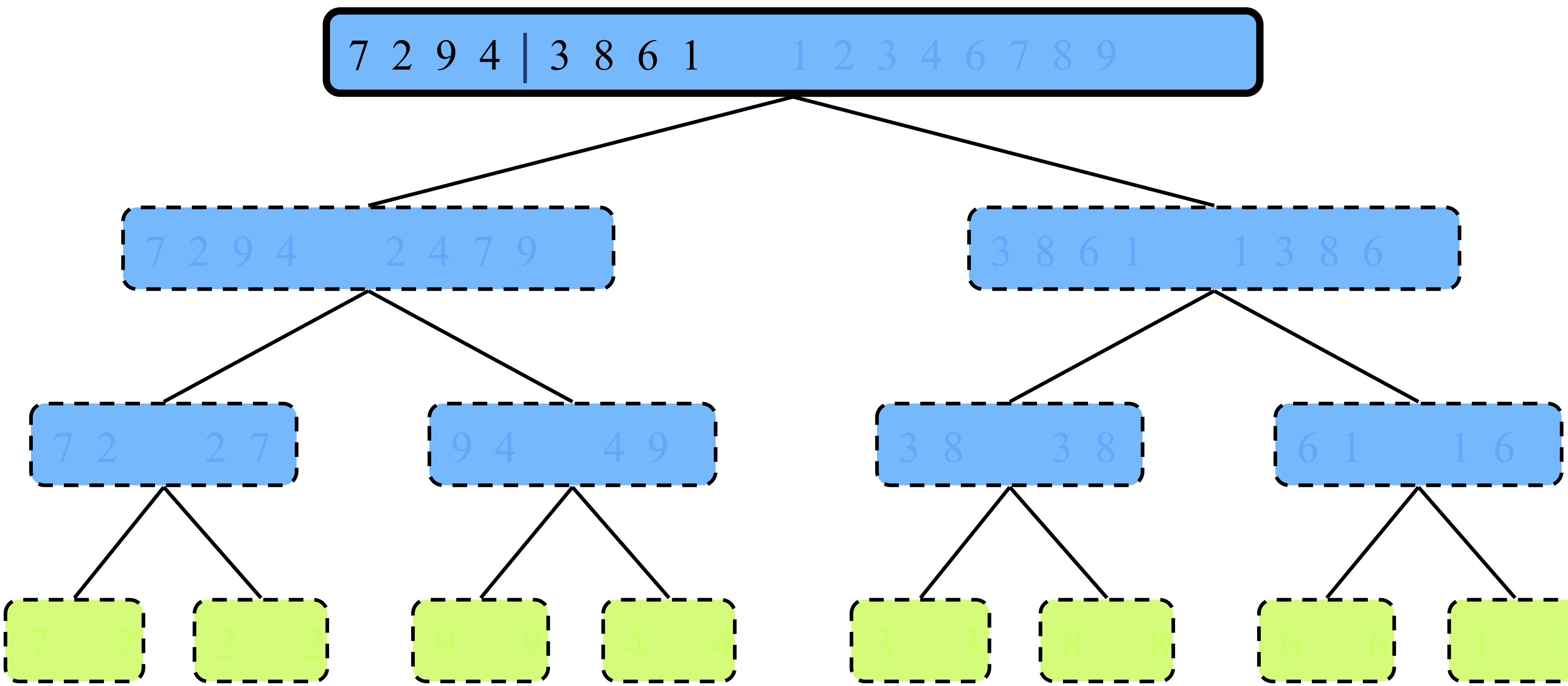
An execution of merge-sort is depicted by a binary tree

- each node represents a recursive call of merge-sort and stores
  - unsorted sequence before the execution and its partition
  - sorted sequence at the end of the execution
- the root is the initial call
- the leaves are calls on subsequences of size 1



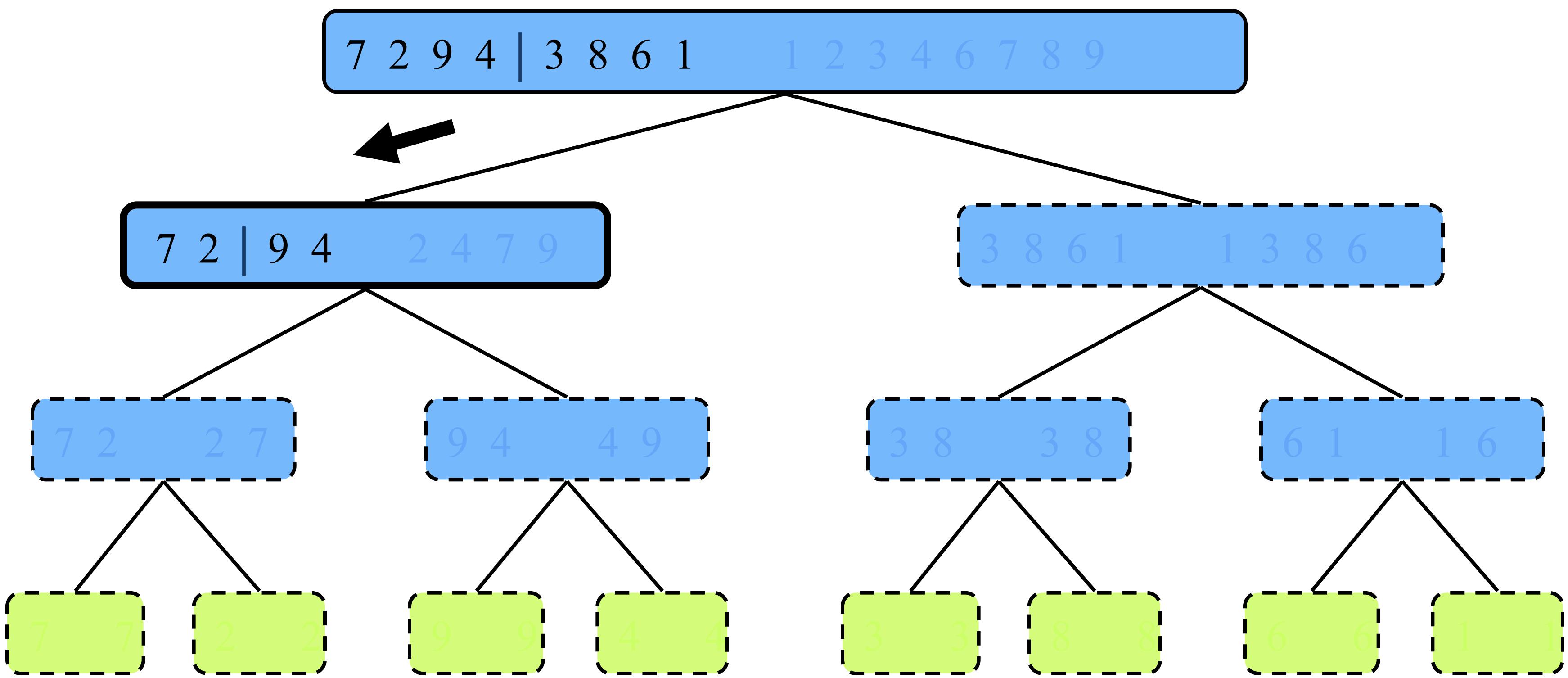
Execution Example

- Partition



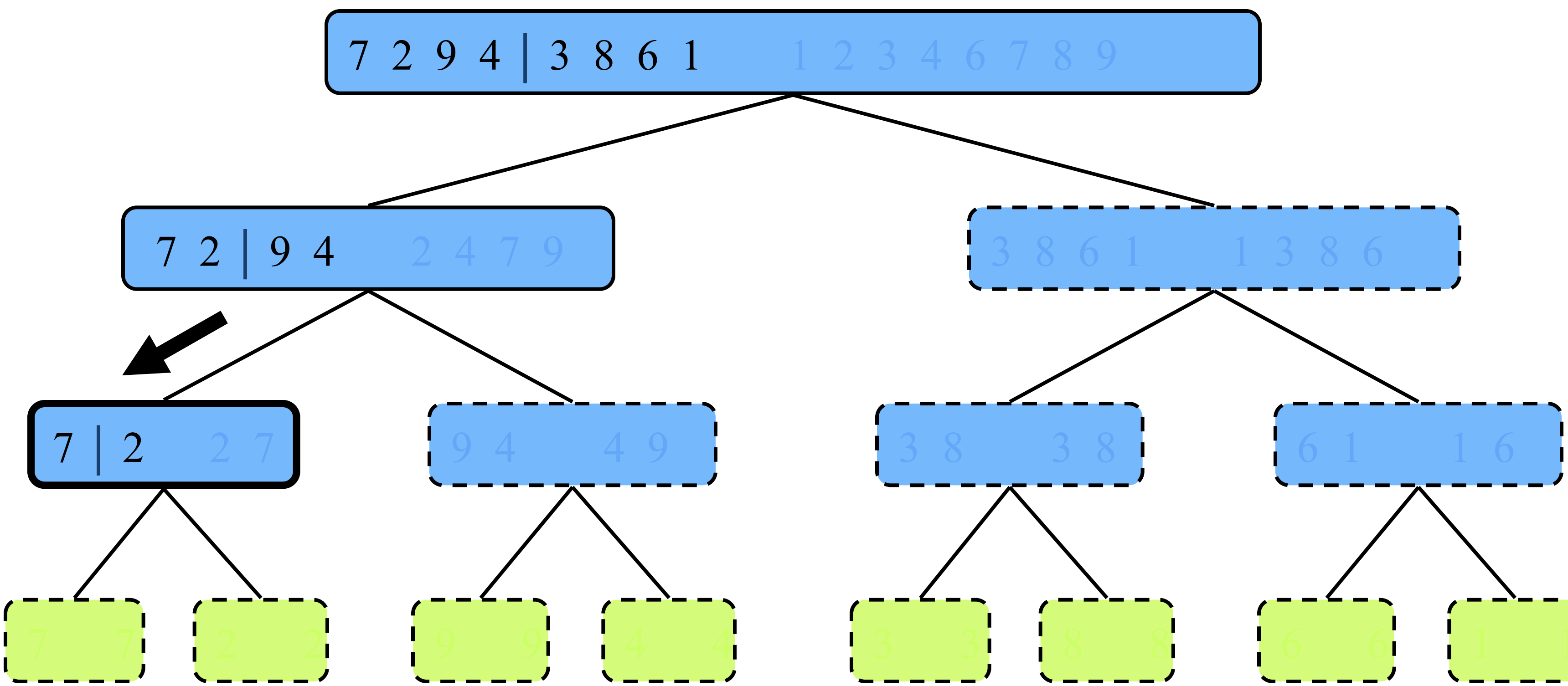
Execution Example (cont.)

- Recursive call, partition



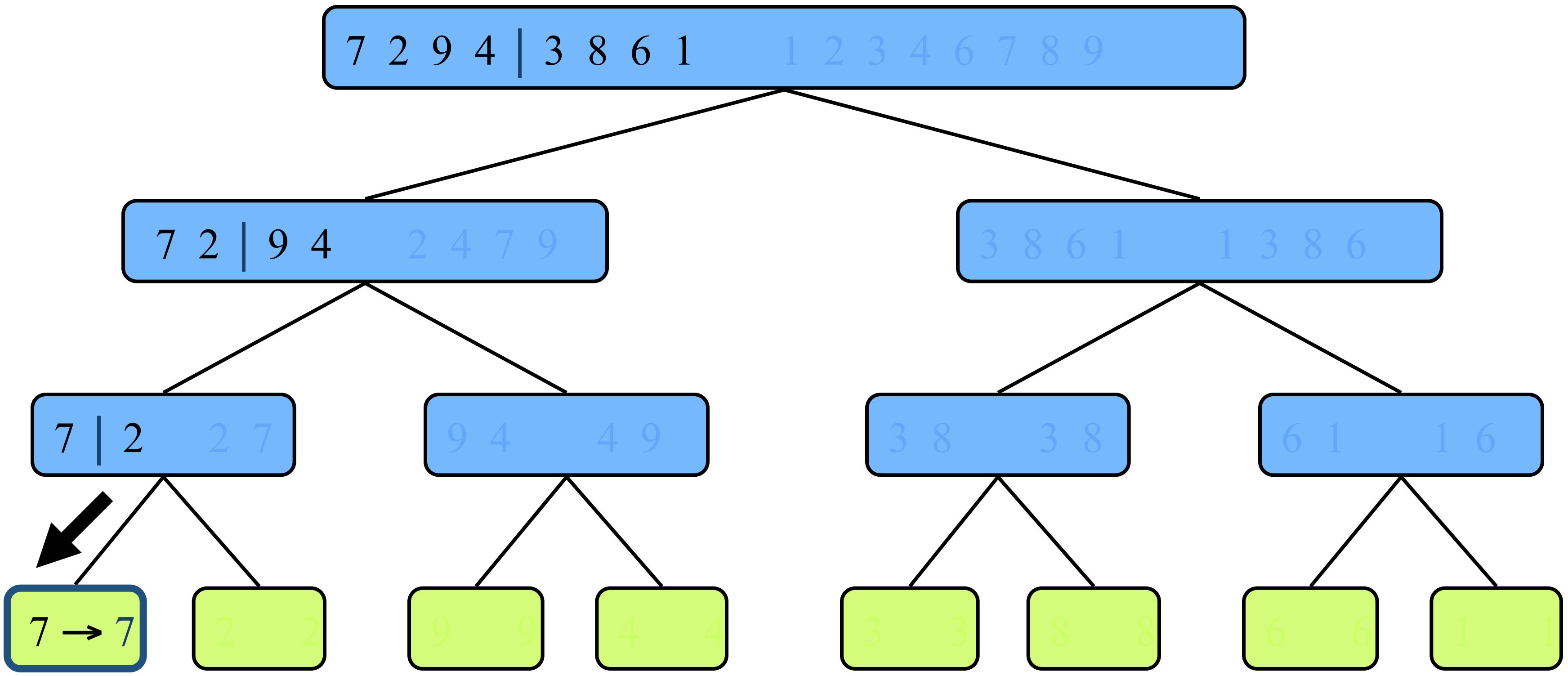
Execution Example (cont.)

- Recursive call, partition



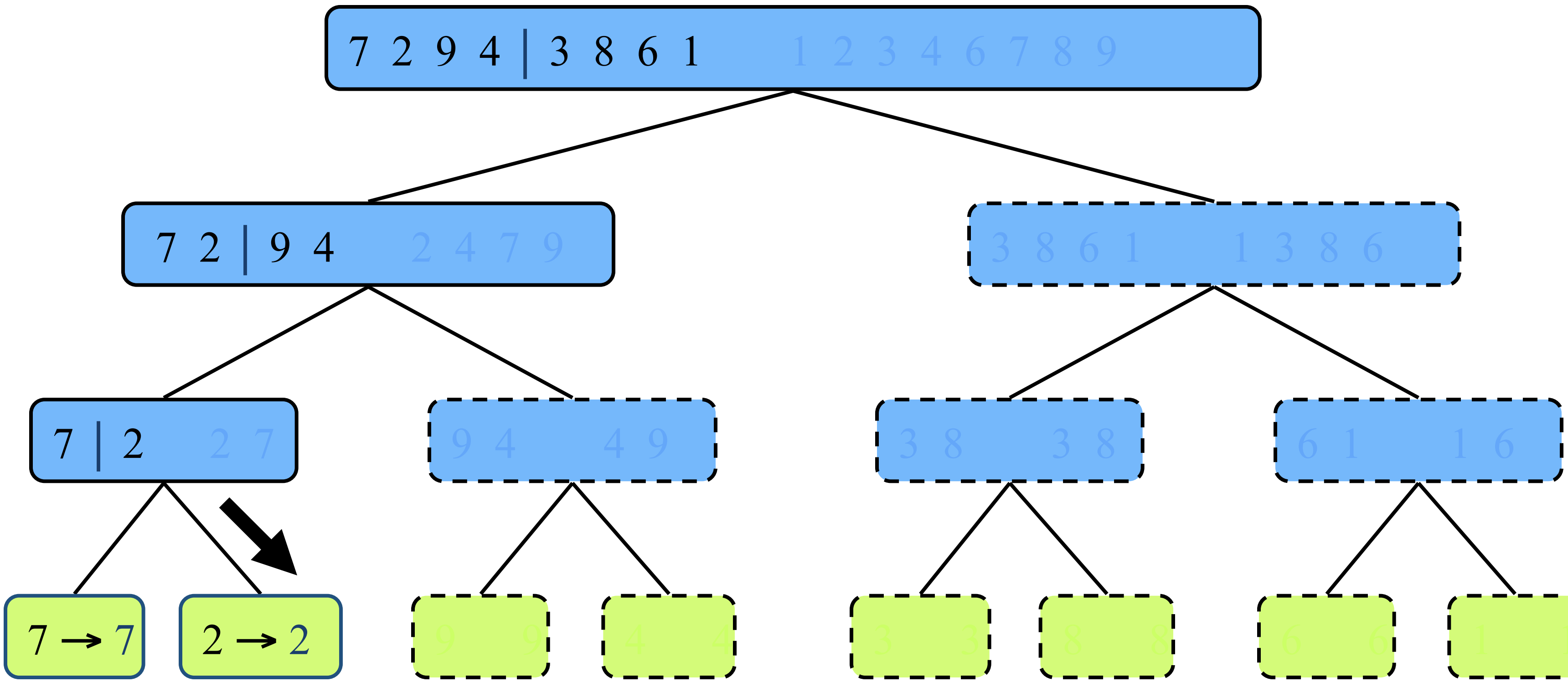
Execution Example (cont.)

- Recursive call, base case



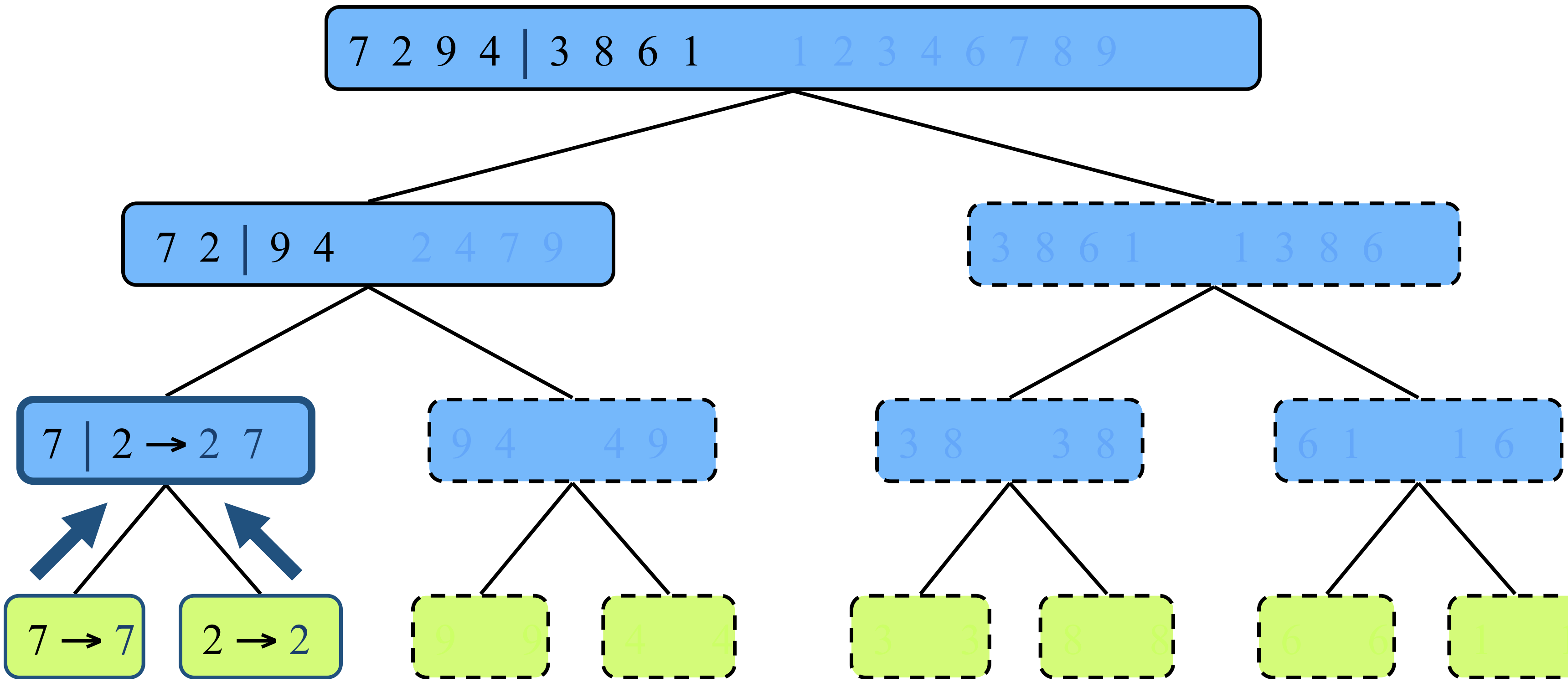
Execution Example (cont.)

- Recursive call, base case



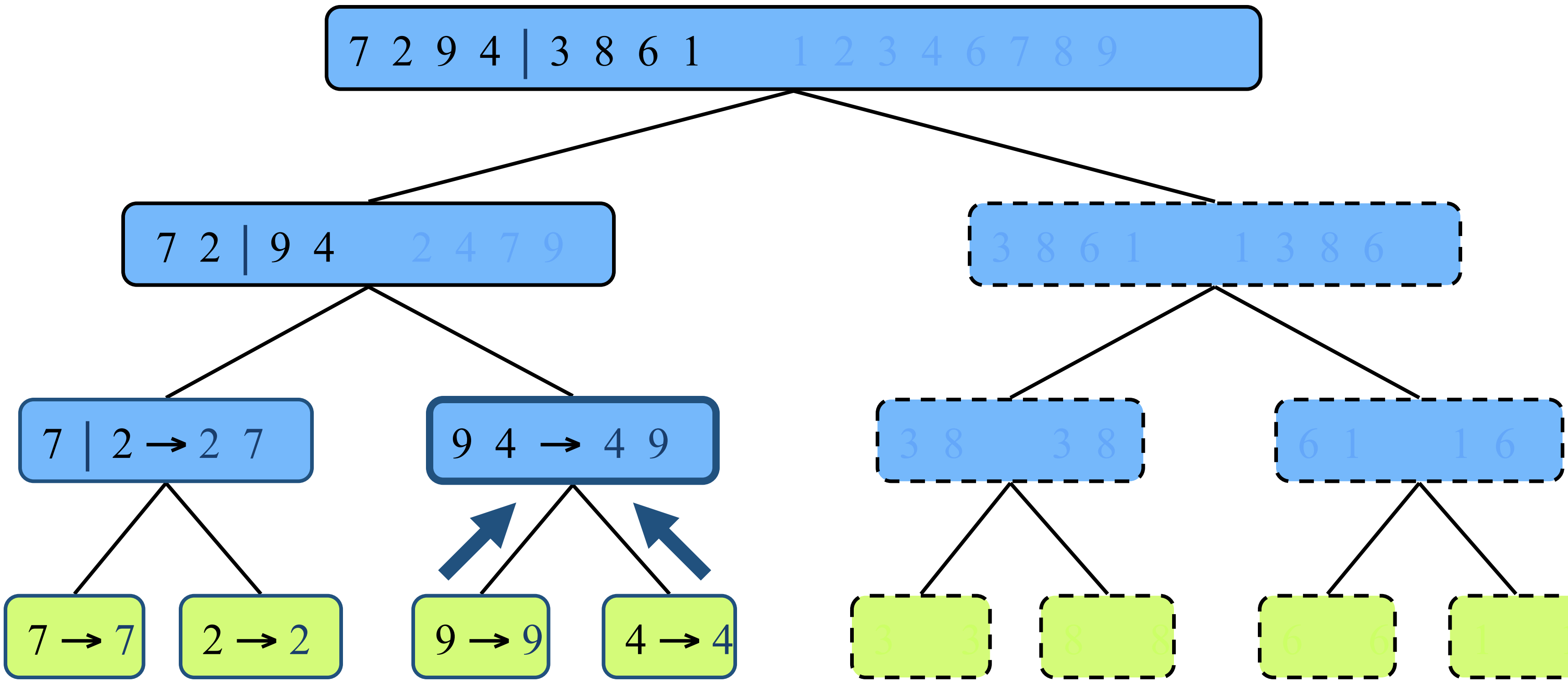
Execution Example (cont.)

- Merge



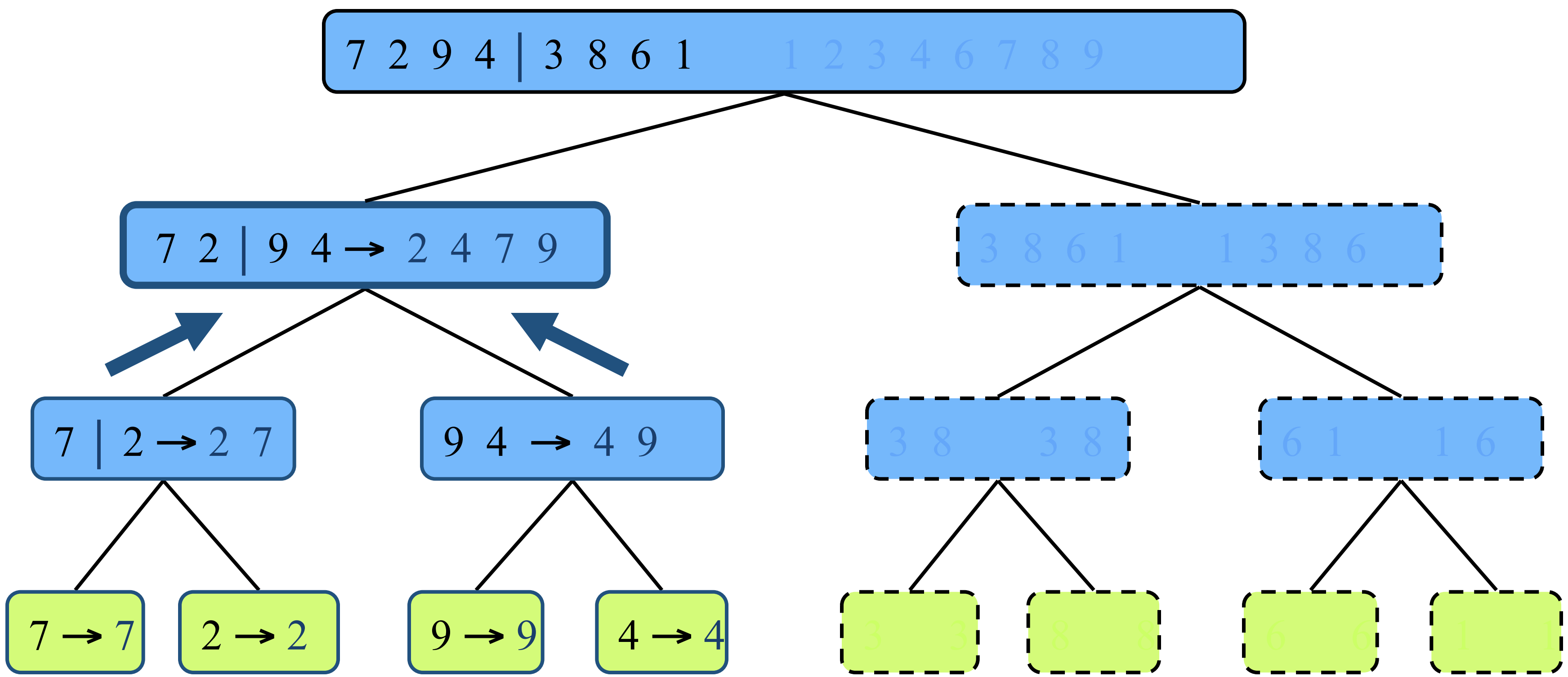
Execution Example (cont.)

- Recursive call, …, base case, merge



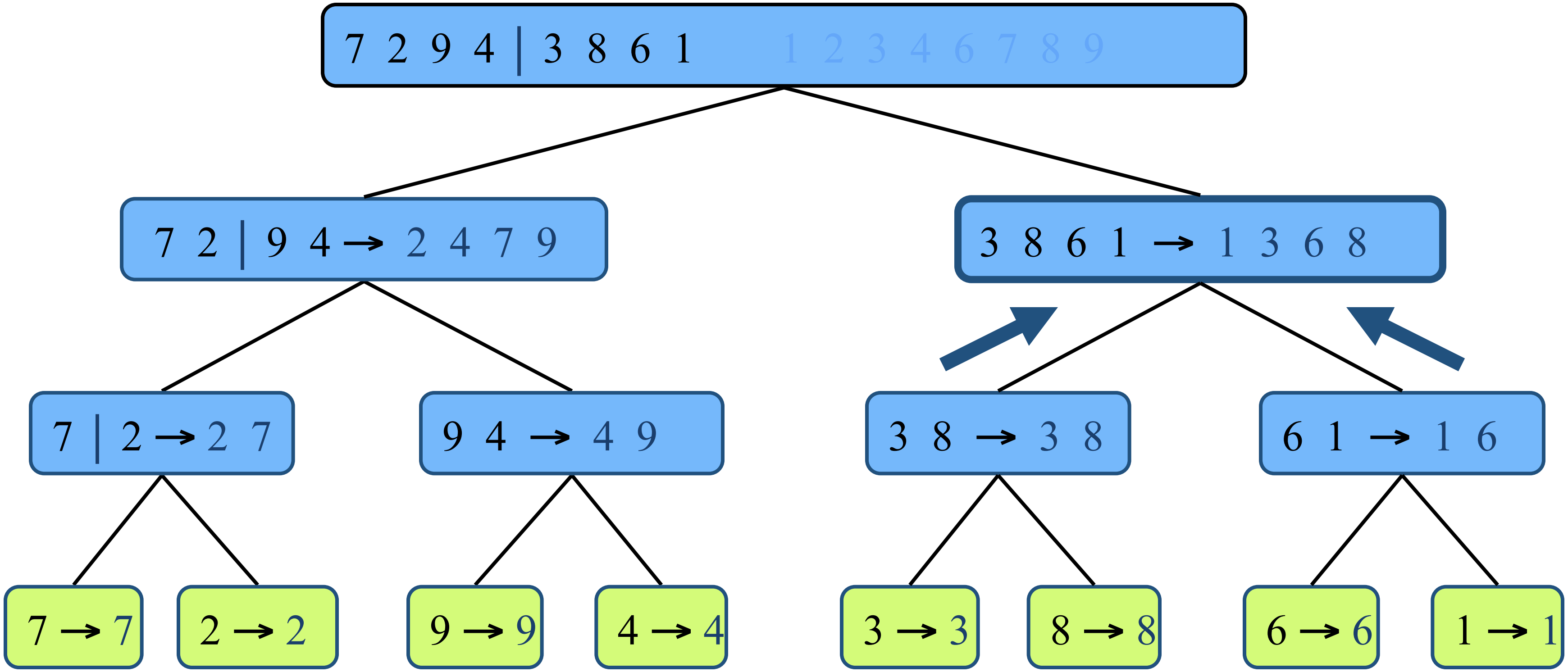
Execution Example (cont.)

- Merge



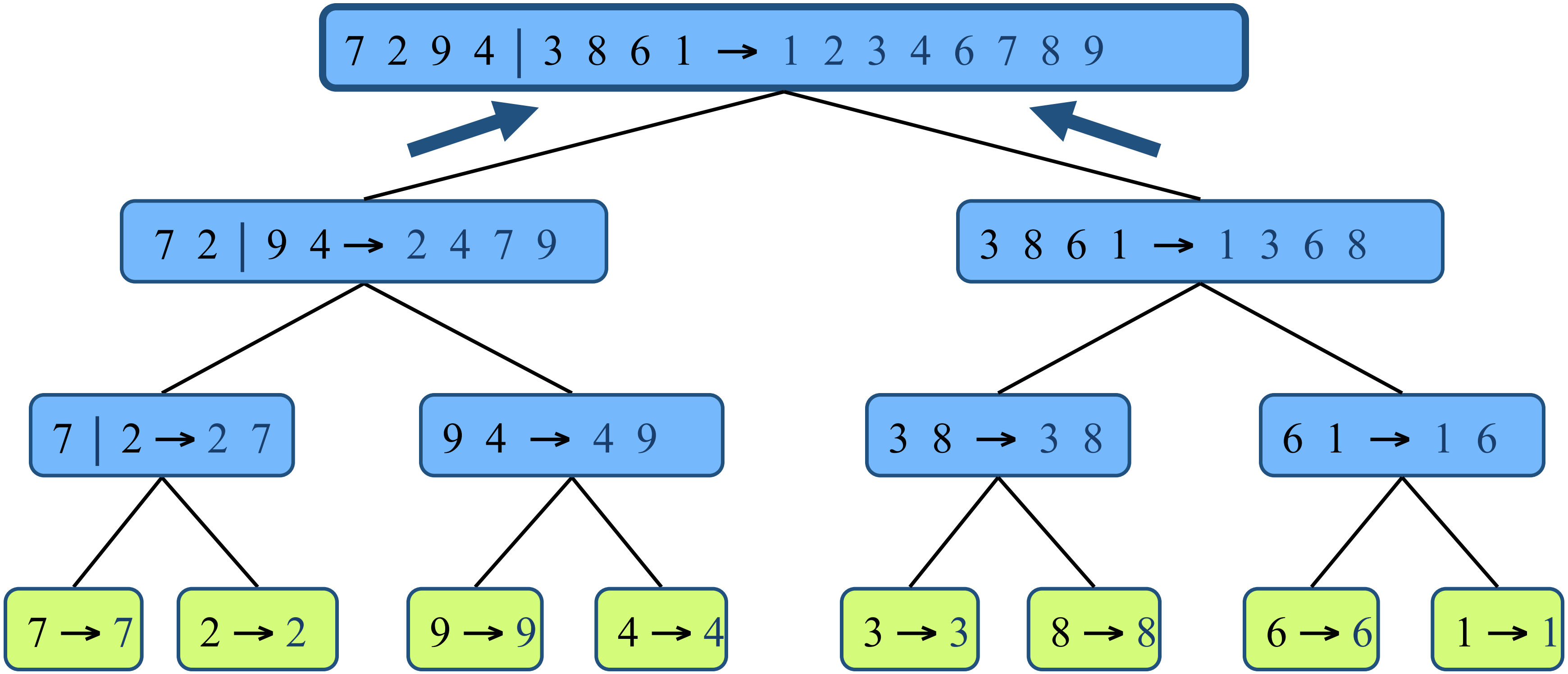
Execution Example (cont.)

- Recursive call, …, merge, merge



Execution Example (cont.)

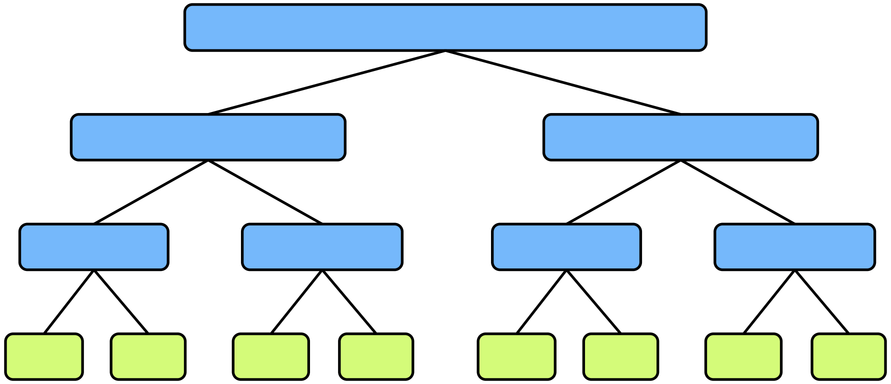
- Merge



Analysis of Merge-Sort

- The height h of the merge-sort tree is $O(\log n)$
  - at each recursive call we divide the sequence in half
- The overall amount or work done at the nodes of depth i is $O(n)$
  - we partition and merge 2^i sequences of size $n/2^i$
  - we make 2^{i+1} recursive calls
- Thus, the total running time of merge-sort is $O(n \log n)$

depth	#seqs	size
0	1	n
1	2	$n/2$
i	2^i	$n/2^i$
…	…	…



Comparing sorting algorithms

Consider the following when evaluating a sorting algorithm:

- Time complexity
- Space complexity
  - An in-place algorithm requires only $n + O(1)$ space, using the already given space for the n elements and $O(1)$ additional space
- Stability
  - A sorting algorithm is stable if it preserves the original relative ordering of elements with equal value
  - Ex: Unsorted sequence (**B**, **b**, a, c). Suppose B = b and a < b < c.
    - Stable sorted: (a, **B, b**, c)
    - Unstable sorted: (a, **b, B**, c)
  - Necessary if we want to sort repeatedly by different attributes (i.e., sort by first name, then sort again by last name)

Summary of Sorting Algorithms

Algorithm	Time	Notes
selection-sort	$O(n^2)$	◆ in-place ◆ not stable ◆ for small data sets (< 1K)
insertion-sort	$O(n^2)$	◆ in-place ◆ stable ◆ for small data sets (< 1K)
heap-sort	$O(n \log n)$	◆ in-place ◆ not stable ◆ for large data sets (1K — 1M)
merge-sort	$O(n \log n)$	◆ not in-place ◆ stable ◆ sequential data access ◆ for huge data sets (> 1M)

Other

- You are given a query point ***p*** and a set ***S*** of ***n*** other points in two dimensional space. Find ***k*** points out of the ***n*** points which are nearest to ***p***.